A potential hierarchy of properties

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Set potentialism

Russell's paradox for sets tells us that there's no set of all and only the non-self-membered sets. In other words, there is no *Russell set r* such that:

$$\forall y(y \in r \leftrightarrow y \notin y)$$

In response, the *set potentialist* claims that although there *isn't* a set of the non-self-membered sets, there *could* be.

More precisely, the claim is that if xx is the plurality of the non-self-membered sets, then although there is no set of the xxs, there could be. Formally:

$$\neg\exists x\forall y(y \in x \leftrightarrow y \prec xx)$$

$$\Diamond\exists x\forall y(y \in x \leftrightarrow y \prec xx)$$

In general, the set potentialist thinks all that's required for the possible existence of a set is the possible co-existence of its members. In a slogan:

possible set existence is a matter of possible co-existence

Examples:

- the people on this Zoom call could form a set
- the objects in our galaxy could form a set
- all of the sets whatsoever could form a set
- all of the non-self-membered sets could form a set

In general:

no matter what things you have, they could form a set

What is the relevant notion of possibility here? The sense in which any things *can* form a set?

There are number of different answers, but the one I'm going to focus on today is that advocated by Øystein Linnebo and James Studd: namely, that it's an *interpretational* notion.

Very roughly:

> *"it's possible that ϕ" means something like "it's possible to reinterpret the language so that ϕ is true"*

So, informally, the set potentialist claims that no matter what possible interpretation I of the language of set theory we consider, and any things in the domain of I, there could be an interpretation I' extending I with a set of those things.

Of course, we don't have to stop at I'. We can take any things in the domain of I' and find a further possible interpretation in which they form a set. And so on.

In general, by iterating this procedure over any possible interpretation, we can generate a rich universe of possible sets. Enough, given the right background assumptions, to satisfy the axioms of set theory: namely, the axioms of ZFC!

We can express the basic idea more precisely and more bloodlessly using a modal operator $\Diamond$ intended to capture the relevant sense in which possible interpretations can be extended to add new sets and backtracking operators $\uparrow$ and $\downarrow$ that allow us to index and access specific such interpretational possibilities.

(Set Collapse$^\Diamond$) $$\uparrow \Diamond \exists s \forall y (y \in s \leftrightarrow \phi \wedge \downarrow Ey)$$

- Note that s's members are restricted to things that "had" existed. Thus, s can be outside of the range of things it collects.
- That's what blocks inconsistency and what allows us to assert the principle for any ϕ whatsoever.
- In other words: Set Collapse$^\Diamond$ operates primarily via *ontological expansion*.

Property potentialism

A version of Russell's paradox also arises for properties. It says that there is no property of being a non-self-applicable property. In other words: there's no *Russell property* r such that:

$$\forall y(y\eta r \leftrightarrow \neg(y\eta y))$$

where $x\eta y$ express the claim that the property y applies to x.

The goal of this talk is to articulate an analogue of set potentialism for properties, in response. The basic idea is all that's required for the possible existence of a property is the possible availability of its definition. In a slogan:

possible property existence is a matter of the possible availability of a definition

Examples:

- since the notion of being self-identical is available in the language we currently speak, there could be the property of being self-identical. Such a property will apply to itself!
- there could be the property of being red
- no matter what we actually mean by "property", there could be a property of being a property in that sense.
- no matter what we actually mean by "application", there could be a property of being non-self-applicable in that sense.

In general:

no matter what notion you have, it could define a property

I'm going to follow the set potentialist in thinking of the relevant notion of possibility interpretationally.

So, informally, my property potentialist will think that no matter what possible interpretation I we consider, and any condition ϕ, there could be an interpretation I' extending I with a property of being ϕ-interpreted-according-to-I.

Of course, we don't have to stop at I'. We can take any new notion in I' and find a further possible interpretation in which it defines a property. And so on.

In general, I'll show that by iterating this procedure over any possible interpretation, we can generate a rich universe of possible properties. In a sense I'll make precise, we can generate enough properties, given the right background assumptions, to satisfy the full second-order comprehension schema!

We can express the basic idea more precisely and more bloodlessly, as we did for sets, using a modal operator $\Diamond$ intended to capture the relevant sense in which possible interpretations can be extended to add new properties and backtracking operators $\uparrow$ and $\downarrow$ that allow us to index and access specific such interpretational possibilities.

(Property Collapse$^\Diamond$) $$\uparrow \Diamond \exists p \Box \forall y \Box (y \eta p \leftrightarrow \downarrow \phi)$$

- Note that p's definition is restricted so as to be a notion that "had" existed. Thus, p's application relation can be outside the range of notions used to define it.
- That's what blocks inconsistency and what allows us to assert that p applies to absolutely any possible object satisfying the definition.
- In other words: Property Collapse$^\Diamond$ operates primarily via *ideological expansion*.

Plan

In the rest of this talk, I'll do two things:

- I'll present a formal theory for property potentialism and outline some results showing that it delivers a rich universe of potential properties.
- Then I'll outline some residual issues.

A formal theory: Language

We start with an interpreted first-order language $\mathcal{L}$, which (for simplicity) I'll assume has a relation symbol R and a predicate symbol N as its non-logical components. The idea is that R and N can have any possible interpretation whatsoever. I'll call $\mathcal{L}$ the **base language**.

Next, we extend $\mathcal{L}$ with some property-theoretic resources: a relation symbol η for property application and a predicate symbol P for being a property. Let $\mathcal{L}_\eta$ be $\mathcal{L} + \{\eta, P\}$.

Finally, we extend $\mathcal{L}_\eta$ with some modal resources: with $\Diamond$, $\uparrow$, and $\downarrow$. Let $\mathcal{L}_\eta^\Diamond$ be $\mathcal{L}_\eta + \{\Diamond, \uparrow, \downarrow\}$.

Logic (I)

I'm going to assume that as we move from one possible interpretation to another, we only add properties to the properties we've already got. So, the interpretations of η and P expand whilst the interpretations of R and N stay constant. Given this way of thinking about $\Diamond$, some principles of modal logic are immediate.

We can assume a positive free quantificational logic together with the minimal normal modal logic K for $\Diamond, \uparrow$, and $\downarrow$.

Since $\Diamond$ concerns *extensions* of a given interpretational possibility, we can also add the T and 4 principles.

(T) $$\phi \to \Diamond\phi$$

(4) $$\Diamond\Diamond\phi \to \Diamond\phi$$

Logic (II)

Finally, we can add the following principles for $\uparrow$ and $\downarrow$, which just lay down some minimal constraints on indexing and accessing possible interpretations.

(1) $$\uparrow \Box \downarrow \phi \leftrightarrow \phi$$

(2) $$\uparrow \Box \forall x (\downarrow \Box \phi \rightarrow \phi)$$

(3) $$\downarrow \neg \phi \leftrightarrow \neg \downarrow \phi$$

(4) $$\uparrow \phi \leftrightarrow \phi$$

where ϕ contains no occurrences of $\downarrow$.

Theory (I)

Given the way that we're thinking about $\Diamond$, we also get some non-logical principles.

We have principles partially capturing the idea that the interpretations of N and R are fixed.

(Strong Stability$_R$)
$$R(x,y) \rightarrow \Box R(x,y)$$
$$\neg R(x,y) \rightarrow \Box \neg R(x,y)$$

and:

(Strong Stability$_N$)
$$N(x) \rightarrow \Box N(x)$$
$$\neg N(x) \rightarrow \Box \neg N(x)$$

Theory (II)

We also have principles partially capturing the idea that the interpretations of η and P only expand.

(Stability$_P$) $$P(x) \to \Box P(x)$$

and:

(Stability$_\eta$) $$P(x) \to (y \eta x \to \Box(y \eta x))$$
$$P(x) \to (\neg(y \eta x) \to \Box\neg(y \eta x))$$

Theory (III)

I want to leave open what populates the domains of the possible interpretations as much as possible. So all I will assume is that no matter what possible interpretation I and possible object o we consider, I can be extended to a possible interpretation with o in its domain. Formally:

($\Diamond$-E) $$\Diamond Ex$$

Theory (IV)

Next, we have two substantial principles. The first is just Property Collapse$^\Diamond$.

(Property Collapse$^\Diamond$) $\uparrow \Diamond \exists p \Box \forall y \Box (y \eta p \leftrightarrow \downarrow \phi)$

where $p, q, r, \ldots$ etc range over properties.

Theory (V)

Second, we have a principle which says that *up to their extension*, the same properties are available in all directions in modal space.

So, suppose we can extend to a possible interpretation I with a property p. Then, the claim is that no matter what other possible interpretation we extend to, it can be further extended to a possible interpretation I'' with a property q that has the same extension according to I'' as p has according to I.

There is a fairly hideous way to say this (I *think*) with a lot of backtracking operators, but instead I'm going to add a consequence of the principle (really, it's a consequence of $\Diamond$-E together with this principle). In order to state this consequence, however, I first need to introduce the language of typed—or second-order—properties over $\mathcal{L}$.

The typed language of properties

We add to $\mathcal{L}$ a stock of second-order monadic variables $X, Y, Z, \ldots$ etc, which take the position of ordinary monadic predicates. So, Xx is well-formed and read "x is X". I'll also take $X = Y$ to be well-formed and read "to be X is to be Y". Call this language $\mathcal{L}^2$.

A translation from $\mathcal{L}^2$ to $\mathcal{L}_\eta^\Diamond$

There is a natural translation from $\mathcal{L}^2$ to $\mathcal{L}_\eta^\Diamond$, which takes objects in $\mathcal{L}^2$ to potential objects in $\mathcal{L}_\eta^\Diamond$ and typed properties in $\mathcal{L}^2$ to potential untyped properties in $\mathcal{L}_\eta^\Diamond$. Formally, where x_X, x_Y, x_Z,... etc is a stock of variables in $\mathcal{L}_\eta^\Diamond$ distinct from x, y, z... etc:

- $(x = y)^\Diamond = x = y$, $R(x, y)^\Diamond = R(x, y)$, and $N(x)^\Diamond = N(x)$
- $(X = Y)^\Diamond = x_X = x_Y$
- $(Xy)^\Diamond = y\eta x_X$
- $^\Diamond$ commutes with the connectives
- $(\exists x\phi)^\Diamond = \Diamond\exists x\phi^\Diamond$
- $(\exists X\phi)^\Diamond = \Diamond\exists x_X(P(x_X) \wedge \phi^\Diamond)$

I'll call $\phi^\Diamond$ the *modalisation* of ϕ.

Theory (VI)

The final principle for my theory is then a version of the G principle of modal logic, but restricted to formulas in the range of the translation.

(G$^\Diamond$) $$P(\vec{x}_X) \to (\Diamond\Box\phi^\Diamond \to \Box\Diamond\phi^\Diamond)$$

where $\phi \in \mathcal{L}^2$ with free second-order variables among $\vec{X}$.

For now, you'll just have to trust me that this is indeed a consequence of the principle I mentioned.

Theory (VII)

Let $\mathrm{PT}^{\Diamond}$ be the theory comprising the preceding principles. So, over the background modal logic, $\mathrm{PT}^{\Diamond}$ comprises Strong Stability$_R$, Strong Stability$_N$, Stability$_P$, Stability$_\eta$, $\Diamond$-E, $\mathrm{G}^{\Diamond}$, and Property Collapse$^{\Diamond}$.

Let $\mathrm{PT}^{\Diamond-}$ be $\mathrm{PT}^{\Diamond}$ minus Property Collapse$^{\Diamond}$.

The gold standard

I take it that for the set potentialist, the gold standard is to show that the axioms of ZFC hold in the potential sets.

Similarly, I take it that, over $\mathcal{L}$, the gold standard for the property potentialist is to show that all of the instances of the second-order comprehension schema in $\mathcal{L}^2$ hold in the potential properties. In other words, that, for any condition ϕ in $\mathcal{L}^2$, the modalisation of

$$(\mathrm{Comp}_{\mathcal{L}^2}) \qquad \forall \vec{x} \forall \vec{X} \exists Y \forall y (Yy \leftrightarrow \phi)$$

is true, where $\phi \in \mathcal{L}^2$ with free variables among $\vec{x}, y, \vec{X}$.

And it turns out this standard can be met in $\mathrm{PT}^{\Diamond}$.

Some results

First, we can show that the modalisations of claims in $\mathcal{L}^2$ behave logically just like those claims. For any sentence ϕ in $\mathcal{L}^2$, we have:

Theorem

ϕ is provable in second-order classical logic (without $\mathrm{Comp}_{\mathcal{L}^2}$) just in case $\phi^\Diamond$ is provable in $\mathrm{PT}^{\Diamond -}$.

Second, we can show that with Property Collapse$^\Diamond$ we can meet the gold standard.

Theorem

$\mathrm{PT}^\Diamond$ proves $\phi^\Diamond$ whenever ϕ is an instance of $\mathrm{Comp}_{\mathcal{L}^2}$.

A potential hierarchy of sets and properties

Moreover, it turns out that $\mathrm{PT}^{\Diamond}$ integrates nicely with set potentialism.

If we interpret N as a predicate for being a set and R as set membership and we add some (*almost*) typical set potentialist principles, including Set Collapse$^{\Diamond}$, then we get a theory that interprets full second-order ZFC modified for urelements (without the axiom of Global Choice)!

Note: in this theory, properties are urelements! So, there will be sets of them just as there are sets of sets. Indeed, whenever there are some properties, there could be a set of them by Set Collapse$^{\Diamond}$.

A model for the potential hierarchy of sets and properties (I)

There are nice simple Kripke models of the combined theory.

For something like a "standard" Kripke model, let κ be an inaccessible cardinal and U a set of κ-many objects to serve as urelements (and so properties). Let $V_\kappa(U)$ be a model of the sets transitively of size less than κ over U, Set the set of sets according to $V_\kappa(U)$, and $\in$ the membership relation for $V_\kappa(U)$.

A world for the model is a tuple $\langle x, Set \cap x, \in \cap x \times x, X, I \rangle$ where x is a transitive set in $V_\kappa(U)$, X is a subset of $x \cap U$ (representing the properties at the world), and I is a relation on $V_\kappa(U)$ whose domain is contained in X (representing the application relation at the world).

A model for the potential hierarchy of sets and properties (II)

A world w accesses another v when v is an *end-extension* of w. So, the objects of w are objects in v, the properties in w are properties in v, and the application relation of v extends the application relation of w without changing the application conditions of w's properties.

Let K be this Kripke model.

A model for the potential hierarchy of sets and properties (III)

Then we have the following theorem.

Theorem

Let ϕ be a sentence in $\mathcal{L}^2_\in$ and w a world in K. Then:

$$w \vDash \phi^\Diamond \qquad \leftrightarrow \qquad \langle V_\kappa(U), V_{\kappa+1}(U)\rangle \vDash \phi$$

Moreover:

Theorem

If we extend K in the usual way to interpret $\uparrow$ and $\downarrow$, then it satisfies the combined theory of potential properties and sets I mentioned.

A deep asymmetry

I want to end by highlighting a deep asymmetry between the treatment of sets and properties on my approach.

The theory $\mathrm{PT}^{\Diamond}$ assumes that R and N are strongly stable: that they don't change their application across possibilities. When interpreted as expressing set membership and sethood, that means $\mathrm{PT}^{\Diamond}$ requires that those notions don't change their application across possibilities.

Not all set potentialists accept this, however. For example, some think that if x doesn't exist, then it isn't a set, even if it *could* be a set.

So, what should we do?

Structuralist set potentialism (I)

There is a general split between two kinds of potentialists: the structuralists and the non-structuralists.

The structuralist set potentialist typically thinks that sets are simply the places in suitable systems, where a system is just some objects together with a relation on those objects. For example, Geoffrey Hellman thinks of sets as the places in systems satisfying the axioms of second-order ZFC.

In terms of possible interpretations, the idea would be that sets are the things in the domains of possible interpretations satisfying those axioms. So, for them, what's a set and what's a member of what is not fixed, but varies with the interpretation.

Structuralist set potentialism (II)

The structuralist will typically think, therefore, that any possible object can be a set. In general, any possible object can play any possible set-theoretic role.

But this effectively rules out the possible existence of a property of being a potential set.

Such a property would have to be the universal property!

Non-structuralist set potentialism (I)

The non-structuralist set potentialist typically thinks that the possible sets are sets in some absolute sense. Although what happens to be a set can vary, what could be a set doesn't. Similarly, for set membership.

In terms of possible interpretations, the idea would be that we only consider possible interpretations whose sets *really are* sets and whose membership relation *really is* membership on those sets.

So, although they don't typically accept versions of strong stability for sethood or membership, they do accept the following weaker claims:

(Coherence$_\in$) $$\Diamond x \in y \to \Box\Diamond x \in y$$

(Coherence$_{Set}$) $$\Diamond Set(x) \to \Box\Diamond Set(x)$$

And it turns out these weaker claims suffice for our main results, as long as we modify the translation schema. (In particular, translating $x \in y$ as $\Diamond(x \in y)$ and $Set(x)$ as $\Diamond Set(x)$).

Coherence (I)

OK, so far so good.

Perhaps there are weaker principles that also suffice together with cleverer translations.

But it seems to me these coherence principles are something of a linchpin. If we have coherence for N and R in the base language, then in $\mathrm{PT}^{\Diamond}$ potential properties allow us to make sense of full second-order comprehension over $\mathcal{L}$.

If we don't have coherence for N and R, however, we can't do this.

Coherence (II)

If that's right, we *need* coherence for N and R.

But it turns out that properties are provably not coherent in $\mathrm{PT}^{\Diamond}$.

Theorem

$\mathrm{PT}^{\Diamond}$ *is inconsistent with:*

(Coherence$_\eta$) $\quad \Diamond x \eta y \rightarrow \Box \Diamond x \eta y$

Coherence (V)

On the view I'm advocating, then, sets must be coherent and properties de-coherent. Put another way: possible sets must be treated non-structurally, whereas possible properties must be treated structurally. Is this asymmetry a problem?

I don't know.

Let me end with some informal conjectures.

So:

Conjecture: If we aren't working with a coherent notion of set—perhaps, because as I sometimes worry, there simply isn't one—then property potentialism of the kind I'm interested in is bound to be **weak**. In particular, it won't be able to deliver enough potential properties to satisfy even all Π^1_1 instances of the typed comprehension schema (over $\mathcal{L}$).

However, even if we're not working with a coherent notion of set:

Claim: There is another way of thinking about potential properties on which get enough potential properties to satisfy the predicative instances of the typed comprehension schema (over $\mathcal{L}$). Indeed, we can get more than that, but I'm not overly excited about that bit more.

Thanks!