Sets as structures

Draft

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1 Introduction

Structuralism is the view that mathematics is about *structures*.[1] According to the orthodoxy, mathematical objects like natural numbers and sets are *places* in structures. In this paper, I want to propose a new idea: namely, that mathematical objects like natural numbers and sets *are* structures. I will focus almost exclusively on the case of sets. So, the proposal is that:

sets are structures

More precisely: the idea is that we should think of sets as well-founded extensional structures with a top element.

Here's the plan. In section 2 I outline the two main kinds of structuralism—eliminative and non-eliminative—and explain what the sets as structures view comes to for each. In section 3 I highlight some reasons to prefer the sets as structures view over the orthodoxy. In section 4 I provide a formal theory for the eliminative sets as structures view and show how it can be used to recapture set theory. Section 5 is a technical appendix.

2 Eliminative and non-eliminative structuralism

Eliminative structuralism is the view that mathematics is about *systems*. Intuitively, a system is just some objects—its *domain*—together with a relation (or relations) on those objects.[2] For example, the people who've read this paper together with the relation "is closer to the Sagrada Familia than" counts as a system (though perhaps not a very interesting one). So, systems need not be composed of mathematical—or, in general, abstract—objects. Eliminative structuralism is consistent with *nominalism*. Nevertheless, systems can "play the role" of mathematical domains. For example, we typically think that the natural numbers satisfy the axioms of second-order Peano arithmetic (PA2). We think, in other words, that the natural numbers together with the relation $\leq$ comprise a PA2 system. And it turns out that there is at most one such system, up to isomorphism: PA2 is *categorical*. This means that we can think of each PA2 system as (up to isomorphism) *uniquely* playing the role of the natural numbers, regardless of what objects populate its domain. The elements of those systems

[1] Perhaps unhelpfully, "structures" here is ambiguous between what in section 2 are called "systems" and "structures".

[2] As we will see, there are a number of ways to make this intuitive characterisation precise. See, especially, the discussion in sections 4.2 and 4.3.

then play the role of particular natural numbers. The eliminativist thus proposes to think of truth in arithmetic as truth in all or any such systems (assuming there's at least one). On this account, to say that the number 7 is thus and so is to say that, in every PA2 system, whatever plays the role of the number 7 is thus and so in that system. Equivalently, to say that the number 7 is thus and so is to say that, in some PA2 system, whatever plays the role of the number 7 is thus and so in that system.

For set theory, the situation is more complicated. The analogue of PA2 for sets is second-order Zermelo-Fraenkel set theory with the axiom of choice (ZFC2). And it turns out that there may be many non-isomorphic ZFC2 systems, none of which is a preferred candidate to play the role of the sets. Unlike PA2, ZFC2 may not be categorical.[3] Nevertheless, it is *quasi-categorical*: any two ZFC2 systems are either isomorphic or one is isomorphic to an inaccessible rank of the other. In other words, any two ZFC2 systems are either isomorphic or one is isomorphic to a V_α, for α an inaccessible cardinal, of the other.[4] This means that even though no particular ZFC2 system plays the role of *all* the sets, each ZFC2 system can be thought of as (up to isomorphism) *uniquely* playing the role of an inaccessible rank in the sets, regardless of what objects populate its domain.[5] The elements of those systems then play the role of particular sets within those ranks.

The quasi-categoricity of ZFC2 tells us that its systems are systematically connected, each building upon the last seemingly towards some ultimate supersystem. So: *is* there such a supersystem? Perhaps a system corresponding to the union of all ZFC2 systems? And if there is, wouldn't that be an ideal candidate to the play the role of all the sets? Yes, and no. We can show, given mild background assumptions, that there will always be such a supersystem. In particular, Tait [1998] proved that quasi-categoricity holds for the theory which extends the axioms of Extensionality and Second-order Separation with the claim that the sets are arranged into ranks.[6] Call this theory R. Whereas ZFC2 systems play the role of inaccessible ranks, R systems play the role of ranks simpliciter. And there is always a maximal R system: that is, an R system to which every R system—and thus, a fortiori, every ZFC2 system—is either isomorphic or isomorphic to one of its ranks.[7] Maximal R systems

[3]If some ZFC2 system contains an inaccessible cardinal, ZFC2 provably *won't* be categorical.

[4]As usual, the V_αs are defined in ZF by transfinite recursion as follows.

- $V_0 = \emptyset$
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ (where $\mathcal{P}(x)$ is the powerset of x)
- $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ (where λ is a limit ordinal)

ZF proves that the V_αs exhaust the sets: every set is in some V_α. I'll use "is a rank" and "is an initial segment" interchangeably with "is a V_α".

[5]Indeed, in ZF, each inaccessible rank satisfies the axioms of ZFC2. So, we can think of the ZFC2 systems as playing the role of all and only the inaccessible ranks.

[6]One way to obtain such a theory is by adding to the axiom of extensionality the second-order claim that there is a collection C of sets well-ordered by membership such that (1) every set is a subset of some set in C, (2) any subcollection of a set x in C forms a set and that set is in every y in C for which $x \in y$, and (3) every set in x in C is a subset of some y in C for $y \in x$. In this theory, each x in C turns out to be a V_α. As a consequence, any system satisfying the theory is either of the form V_λ, for λ a limit ordinal, or of the form $V_{\alpha+1}$ (even though neither λ nor $\alpha + 1$ will strictly speaking exist according to the theory). See Tait [1998] for discussion and for the proof of quasi-categoricity.

[7]To prove this, we need, beyond some standard background assumptions about systems, two principles. The first is a global well-ordering principle says that there is a relation well-ordering all of the objects. The second is a choice principle which says that if φ relates objects to systems, then there is a relations which associates—via coding—one such system to each object. (It is straightforward to formulate these principles

are therefore unique, up to isomorphism. They may or may not correspond to the union of all ZFC2 systems, but even if they don't, that union will uniquely correspond to one of their ranks. Since it ultimately makes no difference to my arguments, I'll assume for simplicity that maximal R systems *do* correspond to the union of all ZFC2 systems.[8]

So why not take maximal R systems to play the role of all the sets? The eliminativist could then think of truth in set theory as truth in all or any such systems. To say that the empty set is thus and so would then be to say that, in every (equivalently: some) maximal R system, whatever plays the role of the empty set is thus and so in that system. There are good reasons for eliminativist not to take this route, however. First, there may, as a matter of fact, fail to be enough objects to constitute even one ZFC2 system. After all, ZFC2 domains comprise an inaccessible infinity of objects, many more than are postulated by modern physics. Second, even if there are enough objects to constitute a single ZFC2 system, there might not be the right number of objects overall for maximal R systems to play the role of all the sets. One nice feature of ZFC2 systems is that, by design, they satisfy all of the standard principles of set theory. But mere R systems need not. They might correspond to successor ranks, for example, or countable ranks, and consequently invalidate the axioms of Powerset or Collection.[9] Finally, even if the maximal R systems satisfy a suitably large fragment of set theory—for example, if they turn out to be ZFC2 systems themselves—it seems overly parochial to take them to play the role of all the sets. It's natural to think that, no matter what objects there are, there could have been more. Indeed, it's natural to think there could have been so many more objects that they constitute genuinely new ZFC2 systems. Generalising, it's natural to think that *necessarily*, no matter what objects there are, there could have been a ZFC2 system with more objects in its domain.[10] Denying that there could be such systems seems to involve

precisely in the plural logic outlined in section 4, or in second-order logic.) *Proof sketch:* Let W be a global well-order. Let $\varphi(x, X)$ relate each x to any R system X whose ordinals are isomorphic to x according to W and otherwise to the empty X. By the choice principle, there will be a Z associating—via coding—one such X for each x. We then use this Z and comprehension principles to recursively build a single R system extending (or isomorphic to) all of the R systems so associated. □

[8] Although ZFC2 systems are the natural analogue of PA2 systems—satisfying, as they do, all of the standard principles of set theory—it turns out that the eliminativist can in principle work with a range of different kinds of system (see Roberts [2019] section 2.7.4 for discussion). In practice, the only other systems that eliminativists have worked with are R systems (see, for example, Putnam [1967] and Berry [2022]). Nevertheless, since it won't significantly change the points I'm going to make, I'll continue to follow Hellman [1989] and Roberts [2019] and focus on ZFC2 systems.

[9] In ZF minus Replacement, the Replacement and Collection schemas are equivalent. Similarly, for the second-order Replacement and Collection axioms in ZFC2. For my purposes, it turns out to be more instructive to work with Collection. So, in what follows, I will assume that ZF has been formulated with a schema of Collection. Similarly, for ZFC2.

[10] The *necessitist* denies this (see Williamson [2013]). For them, existence is a modally invariant matter and any system that could exist, does exist. Any maximal R system will consequently be a necessarily maximal R system and have all possible R and ZFC2 systems as ranks, up to isomorphism (assuming that systems don't change their structure across possibilities where they exist (see the stability axioms in section 4.2 and [Roberts, 2019, p.828-9])). A central challenge for the necessitist eliminativist is to explain why maximal R systems—or one of their ranks—satisfy enough set theory to play the role of all the sets. In any case, since most eliminativists reject necessitism and since the most interesting issues and challenges arise when it is rejected, I will assume for the rest of this paper that necessitism is false. That is, I will assume *contingentism*, the view that existence is not a modally invariant matter. (Some care is needed on the terminology here. Many philosophers take metaphysical possibility to be governed by the S5 system of modal logic, and eliminativists have typically assumed that S5 also governs their preferred notion of possibility (see, for example, [Hellman, 1989, p. 19], [Hellman, 1990, p.427], and [Berry, 2022, p.49]). In S5, the claim that first-order existence is a modally invariant matter can be formalised as the claim that necessarily, every object necessarily exists:

an arbitrary and unjustified curtailing of possibility. But if there could have been new ZFC2 systems, on what basis can we legitimately ignore them? What's so special about the systems there happen to be, rather than the systems there merely could be? What if it turned out that the generalised continuum hypothesis is false, but only in merely possible ZFC2 systems? Or, there were measurable, or Woodin, or whatever, cardinals, but only in merely possible ZFC2 systems? Ignoring such systems seems to involve an arbitrary and unjustified curtailing of Cantor's paradise.

In response, the eliminativist typically expands the scope of their view to include merely possible ZFC2 systems.[11] And it turns out that even though there may be no possible system that can play the role of all the sets—perhaps because, no matter what systems there could be, there could always be larger ZFC2 systems—the eliminativist can make use of a clever trick to talk *as if* there were such systems.[12] The idea is to simulate talk about what's true in the union of all possible ZFC2 systems by talking about what's true in specific, though ever more inclusive, such systems. An example will help to illustrate. Consider the claim that every set is a member of some set. The thought is that this claim is intuitively true in the union of all possible ZFC2 systems if, and only if, necessarily, for any ZFC2 system S and any object o in its domain, there could be a ZFC2 system S' *end-extending* S containing an object o' such that according to S', o is a member of o'.[13] Formally:

$$\Box\forall S\forall x \in S\Diamond\exists S' \sqsupseteq S\exists y \in S'(S' \vDash x \in y)$$

where $S' \sqsupseteq S$ abbreviates the claim that S' is an end-extension of S and $S' \vDash x \in y$ abbreviates the claim that according to S', x is a member of y.

In general, we have a systematic way of translating claims ostensibly about the sets to claims about what's true in ever increasing possible ZFC2 systems and thus intuitively what's true in the union of all possible ZFC2 systems. It's called the *Putnam-translation* schema, and defined by recursion, as follows (where S, S', S'',... etc range over ZFC2 systems).[14]

- $(x \in y)^{pt}_S = S \vDash x \in y$

$\Box\forall x\Box Ex$ (NNE). In various weaker modal logics, NNE leaves open that there could be more objects than there in fact are. In terms of Kripke models: NNE ensures that we don't lose objects as we move along the accessibility relation, but it allows that we might gain objects. In such logics, the claim that first-order existence is a modally invariant matter is best formalised as the combination of NNE and the Barcan Formula, which is the schema saying, for each φ, that if there could be a φ, then something could be a φ: $\Diamond\exists x\varphi \to \exists x\Diamond\varphi$ (BF). Call NNE + BF *strong (first-order) necessitism* and NNE *weak (first-order) necessitism*. In S5, strong and weak necessitism are equivalent, whereas in weaker systems, they need not be. It is strong necessitism and its extension to systems—in particular, BF and its extension to systems—together with assumptions guaranteeing that systems don't change their structure across possibilities where they exist which ensures the systems there are effectively comprise all the systems there could be and thus that any maximal R system is a necessarily maximal R system. So, strictly speaking, it is strong necessitism—and in particular, BF and its extensions—that I am here, and for the rest of the paper, assuming is false.)

[11] Similarly, for PA2 systems and arithmetic.

[12] See Putnam [1967], Hellman [1989], and Roberts [2019].

[13] A system S' end-extends a system S if S is a subsystem of S' and S's sets are transitive in S'. More precisely, S' end-extends S if (i) S and S' both exist, (ii) S's domain is included in S''s, (iii) for any o, o' in S's domain, $S \vDash o \in o'$ if and only if $S' \vDash o \in o'$, and (iv) for any o in the domain of S and o' in the domain of S', if $S' \vDash o' \in o$, then o' is in the domain of S.

[14] See Roberts [2019] for a precise formulation.

- $(x = y)^{pt}_S = x = y$
- ^{pt} commutes with the connectives
- $(\exists x\varphi)^{pt}_S = \Diamond\exists S' \sqsupseteq S \exists x \in S' \varphi^{pt}_{S'}$
- When φ is a sentence, φ^{pt} abbreviates $\varphi^{pt}_{\emptyset}$

The eliminativist thus proposes to think of truth in set theory as truth of the corresponding Putnam-translation. On this account, to say that the empty is thus and so is to say that, in every possible ZFC2 system S, whatever plays the role of the empty set in S is (thus and so)$^{pt}_S$.

For Putnam-translation to succeed in providing an interpretation of set theory, two minimal constraints need to be met. First, the Putnam-translations of logical truths must come out true. In particular, the Putnam-translations of instances of the vacuous quantifier axiom—$\varphi \leftrightarrow \forall x\varphi$ (where x is not free in φ)—must come out true. Formally:

(Stability) $$[\forall\vec{y}(\varphi \leftrightarrow \forall x\varphi)]^{pt}$$

where φ's free variables are among $\vec{y}$. Although Stability may seem innocuous, it articulates a substantial demand on modal space. It captures the cash value for Putnam-translation of the claim that each possible ZFC2 system has access to the very same sets (up to isomorphism): that, for any two possible ZFC2 systems S and S', if o is an element of S, then S' must have a possible ZFC2 end-extension containing an isomorphic copy of o (according to S). Intuitively, then, it captures the thought that truth in the union of all possible ZFC2 systems is indeed equivalent to Putnam-translation. In section 3.3, we'll see that, as a result, Stability may be entangled with a number of tricky metaphysical issues.

The second constraint is that the Putnam-translations of the most basic of axioms of set theory must come out true. At the very least, the Putnam-translation of the axiom of Empty Set must come out true. It's easy to see that the Putnam-translation of the axiom of Empty Set is equivalent to the claim that there could be at least one ZFC2 system. Formally:

(Existence) $$\Diamond\exists S(S = S)$$

Perhaps surprisingly, Stability and Existence already suffice to prove the Putnam-translations of a large fragment of set theory. Let BT be the background theory for possible systems outlined in section 4.2,[15] and let Z^* be ZFC minus the axiom of Collection plus the claim that every set is in some V_α: that is, $\forall x\exists\alpha(x \in V_\alpha)$. Then we have the following theorem.

Theorem 1. *If* Z^* *proves* φ, *then* BT + Stability + Existence *proves* φ^{pt}.[16]

[15] Briefly: over a positive free S5 modal logic, BT contains the most basic of assumptions concerning the existence and stability of systems. In particular, it contains axioms which tell us (1) that, for each $\varphi(x)$ and $\psi(x, y)$, there is a system whose domain is the φs and whose relation coincides with $\psi(x, y)$ and (2) that systems don't change their structure across possibilities where they exist. In assuming that possibility is governed by an S5 modal logic, I follow extant modal eliminativists like [Hellman, 1989, p. 19], [Hellman, 1990, p.427], and [Berry, 2022, p.49] (see also Roberts [2019]). Indeed, the only fully worked out versions of modal eliminativism I know of are formulated over S5. Of course, not everyone accepts its axioms, and it's an interesting question how the view should be formulated in various weaker modal logics; similarly, how some of the problems I raise for the view—especially those in section 3.3—play out in such logics. But these are really questions for another paper. My hope is that my results—together with those in Roberts [2019]—will serve as a solid foundation for answering them. Nevertheless, I'll make some comments on these questions as I go, mostly in footnotes.

[16] We can actually prove something stronger, namely:

Eliminativists usually go beyond Stability and Existence by positively embracing the idea that any possible ZFC2 system can be end-extended by a larger such system.[17] Formally:

(Extendibility) $$\Box\forall S\Diamond\exists S'(S \sqsubset S')$$

Given Extendibility, we can sharpen theorem 1. Let MSST—for *Modal Structural Set Theory*—be BT + Stability + Existence + Extendibility, and let In be the claim that the inaccessible cardinals are unbounded in the ordinals: that is, the claim that above every ordinal there's some inaccessible cardinal. Then we have the following theorem.[18]

Theorem 3. MSST *proves* φ^{pt} *if and only if* $\mathsf{Z}^* + \mathsf{In}$ *proves* φ.

As is standard, I'll refer to eliminativists who expand the scope of their view in include merely possible systems as *modal structuralists*, and I'll refer to those who adopt Putnam-translation and MSST as *orthodox* modal structuralists.

Non-eliminative structuralism is the view that mathematics is about *structures*, where a structure is 'the abstract form of a system' [Shapiro, 1997, p.73]. So, structures are abstracta. The *places* in a structure are the parts that correspond to elements in the domain of the abstracted system. Roughly speaking, they are the abstract forms of those elements. So, places are also abstracta. For the non-eliminativist, structures don't merely play the role of a mathematical domain: the relevant structure and its places *are* that domain. For example, the structure corresponding to any PA2 system *is* the domain of arithmetic; its places *are* the natural numbers. Although there may be many PA2 systems, there is a single natural number structure: namely, the abstract form of all those systems. And although there may be many things that play the role of the natural number 7 in these various systems, there is a single natural number 7: namely, the 7th place in the natural number structure. The non-eliminativist thus proposes to think of truth in arithmetic as truth in the natural number structure. The language of arithmetic therefore gets its "face value" interpretation. In particular, its quantifiers really range over the natural numbers. It's just that the natural numbers, for the non-eliminativist, are the places in the natural number structure. The availability of a face value interpretation is one of the main reasons to prefer eliminativism over non-eliminativism.[19]

Theorem 2. *If* φ *is provable in* ZFC *and* Z^*+ "there are unbounded inaccessible cardinals"*, then* φ^{pt} *is provable in* BT + Stability + Existence.

Theorems 1 and 2 are corollaries of theorem 3 below. *Proof:* It's not hard to see by a simple induction on the complexity of φ that if S cannot be end-extended by a larger system—that is, if S's only end-extensions have the same domain—then φ^{pt}_S just in case $S \vDash \varphi$ (where φ's parameters are in S). Let φ be a sentence provable in ZFC. Since S satisfies all of the axioms of ZFC, it follows that φ^{pt}_S and so $(\exists x\varphi)^{pt}$. Thus, by Stability, φ^{pt}. So, if there could be such an S—in other words, if the Extendibility principle below is false—then φ^{pt}. Putting that together with the left-to-right direction of theorem 3, we get theorems 1 and 2. □

[17] This is related to, but importantly different from, the thought discussed above that necessarily, no matter what objects there are, there could be a ZFC2 system with more objects in its domains. That thought implies that for any possible ZFC2 system S, there could be a larger one. But it doesn't imply that S can have any proper end-extensions, as required by Extendibility. Conversely, Extendibility does not imply that no matter what objects there are, there could have been more. Indeed, Extendibility is strictly speaking consistent with necessitism. See [Roberts, 2019, p. 827-8] for discussion.

[18] See Roberts [2019] section 2.6 and remark 5.25 for details.

[19] See, for example, Shapiro [1997] and Shapiro [2000].

For set theory, the account is analogous. Although there may be many ZFC2 systems, there is a single set structure.[20] And although there are many things that play the role of the empty set in these various systems, there is a single empty set: namely, the 1st place (as it were) in the set structure. Truth in set theory is thus thought of as truth in the set structure, and the language of set theory gets its face value interpretation. In particular, its quantifiers really range over the sets. It's just that the sets, for the non-eliminativist, are the places in the set structure. Given the jump in complexity from the eliminativist's interpretation of arithmetic to their interpretation of set theory via Putnam-translation, the comparative benefit of a face value interpretation is correspondingly greater.[21]

Non-eliminative structuralism faces an immediate and well-known challenge: namely, to explain how it differs from bog-standard platonism. After all, it says that there are abstract objects—sets—and those objects bear a relation—membership—to each other in such a way that the axioms of set theory are true. The standard response to this challenge rests on a specifically structuralist claim about the *nature* of those objects. As Shapiro [2000] puts it:

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other. (p.258)

Similarly, the empty set is no more and no less than the first place in the set structure. No sets have independence from the set structure, and as places in the set structure, none is independent of any other.

I'll refer to non-eliminativists who take set theory to concern the set structure and adopt these dependency claims as *orthodox* non-eliminativists.

Eliminative and non-eliminative structuralism each give rise to their own version of the sets as structures view I want to advocate. For the eliminativist, the proposal is that the role of individual sets should be played not by whatever populates the domains of the various possible ZFC2 systems, but rather by certain systems themselves. In particular, the proposal is that the role of individual sets should be played by possible well-founded extensional systems with a top element. More precisely, let S be a system with domain D and relation R. Intuitively, S is well-founded if there are no infinite descending chains of R relations on D: that is, if there's no infinite sequence of $x_0, x_1, x_2, ...$ of things in D for which $x_1 R x_0, x_2 R x_1, x_3 R x_2, ...$ Formally, we say that S is well-founded if it satisfies the axiom of second-order Foundation.[22]

(Foundation$_2$) $$\forall X(\exists x Xx \rightarrow \exists x(Xx \wedge \forall y \in x(\neg Xy)))$$

[20]The non-eliminativist may take this to be the abstract form of the maximal R systems, but they need not. For example, if there happened to be sufficiently many more non-sets than sets, the identification would fail. In particular, if there were κ-many sets and at least 2^κ-many non-sets, there would be an R system larger than any R system isomorphic to the sets. In that case, the non-eliminativist would face a tricky question: namely, what V_α of the maximal R systems is *the* set structure the abstract form of, and why?

[21]The worries I raised for the non-modal eliminativist don't immediately arise for the non-eliminativist. Since structures and their places are abstracta, we need not worry that there might fail to be enough objects to constitute the set structure (unless we take an Aristotelian view according to which the structure can't exist without being the abstract form of some concrete structure (see Linnebo and Pettigrew [2014] for discussion)). Similarly, we need not worry that there could be more sets than there are. It is standard to assume that abstracta exist of necessity, if at all (though see the discussion in 3.2). So, any set there could be, there is. And every set there is, is in the set structure according to the non-eliminativist.

[22]See Definition 6 in section 4 for a precise definition of satisfaction.

Intuitively, S is extensional if elements of D are identical when the same things R-relate to them. Formally, we say that S is extensional if it satisfies the axiom of Extensionality.

(Extensionality) $$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$$

Intuitively, S has a top element if some element of D is a finite chain of R relations away from every other element of D. Formally, we say that S has a top element if it satisfies the following second-order axiom.

(Top) $$\exists x \forall X(Xx \wedge \forall y, z(Xy \wedge z \in y \to Xz) \to \forall y Xy)$$

Call systems that satisfy $\mathsf{Foundation}_2$, Extensionality, and Top *set-systems*. The proposal is thus that the role of individual sets should be played by possible set-systems.

Identity and membership for sets have natural analogues for set-systems. In particular, given some caveats to be discussed in section 4.3, identity corresponds to possible isomorphism between set-systems, and membership corresponds to the relation one set-system bears to another when the first is possibly isomorphic to the set-system we get by restricting second to some element below its top element. So the idea is that truth in set theory should be thought of as truth in the possible set-systems under those relations. Formally, we have the following translation schema (where $S, S', S'', ...$ etc range over set-systems, and $=^*$ and $\in^*$ abbreviate the analogues of identity and membership).

- $(x \in y)^{tr} = S \in^* S'$
- $(x = y)^{tr} = S =^* S'$
- ^{tr} commutes with the connectives
- $(\exists x \varphi)^{tr} = \Diamond \exists S \varphi^{tr}$

For the non-eliminativist, the proposal is that we should think of sets not as places in the set structure, but rather as the abstract forms of set-systems. Call such structures *set-structures*. Identity between sets then corresponds to identity between set-structures, and membership corresponds to the relation one set-structure bears to another when the first is the abstract form of the set-system we get by restricting second to some place immediately below its top place. So the idea is that truth in set theory should be thought of as truth in the set-structures under those relations. Formally, we have the following the following translation schema (where $s, s', s'', ...$ etc range over set-structures, and $\in^*$ abbreviates the analogue of membership).

- $(x \in y)^t = s \in^* s'$
- $(x = y)^t = s = s'$
- ^{t} commutes with the connectives
- $(\exists x \varphi)^t = \exists s \varphi^t$

In the rest of the paper, I'm going to do two things. First, I'll outline some reasons to prefer the sets as structures view over the orthodoxy. Nothing I say will be conclusive, but it should give you a good sense of the *kind* of work the sets as structures view can do that the orthodoxy can't. Second, I'll provide a formal theory for the eliminativist sets as structures view and show how it can be used to recover ZF via the above translation.

3 In favour of sets as structures

In this section, I'll outline some reasons to prefer the sets are structures view over the orthodoxy.

3.1 Dependence and the iterative conception of set

Recall that in order to distinguish their view from bog-standard platonism, the orthodox non-eliminativist appeals to a claim about the nature of sets: namely, that sets are dependent on the set structure and on each other. Linnebo [2008], however, has argued that this claim is incompatible with the widely accepted *iterative conception of set*, according to which sets are "formed from" their members. As he puts it:

> The relation between a set and its elements is thus asymmetric [according to the iterative conception], because the elements must be 'available' before the set can be formed, whereas the set need not be, and indeed cannot be, 'available' before its elements are formed. A set thus appears to depend on its elements in a way in which the elements do not depend on the set. [Linnebo, 2008, p.72]

He goes on:

> This asymmetric dependence is in fact a very good thing, as there are all kinds of difficult questions about the higher reaches of the hierarchy of sets. How far does the hierarchy extend? Are the different stages rich enough for the continuum hypothesis to fail? It would be a pity if very simple sets, such as the empty set and its singleton, depended on the entire hierarchy of sets, and their identities could therefore not be completely known before these hard questions had been answered. But fortunately the situation is the reverse. In particular, we can give an exhaustive account of the identity of the empty set and its singleton without even mentioning infinite sets. [Linnebo, 2008, p.73]

Linnebo's conclusion: non-eliminative structuralism may be good for some areas of mathematics, but not set theory.

The sets as structures view allows for a response. If s is a set-structure, then according to the non-eliminativist, its places will all depend on one another and on s. But there's no pressure to think that s itself depends on any other set-structure. In particular, if $s_\emptyset$ is the set-structure corresponding to the empty set and $s_{\{\emptyset\}}$ is the set-structure corresponding to the singleton of the empty set, there's no reason to think that $s_\emptyset$ depends on $s_{\{\emptyset\}}$. We can give an exhaustive account of the identity of $s_\emptyset$ without invoking $s_{\{\emptyset\}}$, let alone any infinite sets.

What about the other direction? Does $s_{\{\emptyset\}}$ depend on $s_\emptyset$? Here things are less clear. The empty set seems to be a constituent part of its singleton in a way that $s_\emptyset$ is not a constituent part of $s_{\{\emptyset\}}$.[23] Nevertheless, we may be able to get much of the benefit of the iterative conception without such dependencies. For familiar Russell-like reasons, the non-eliminativist

[23]Of course, $s_\emptyset$ is the abstract form of a subsystem of $s_{\{\emptyset\}}$, and thus the abstract form of a constituent part of $s_{\{\emptyset\}}$.

must reject the naive principle that any system determines a structure.[24] So, they're faced with a very general and fundamental question: when *do* systems determine structures? The challenge is to provide an account of the dividing line between those systems that determine structures and those that don't which explains why there are many of the structures there are—enough for mathematics—but not the problematic structures. One obvious and natural account is based on an iterative conception of structures! We start at the very first stage with no structures whatsoever. We then take any available systems and form the corresponding structures. Since there is the system with no elements in its domain at the first stage, we will have the empty structure at the second stage. We then take any systems of things at the second stage and form the corresponding structures at the next stage. Since there is a system with one element and an empty relation at the second stage, we have $s_\emptyset$ at the third stage. And so on. A system will then determine a structure according to the view precisely when it is isomorphic to some system available at some stage.

New set-structures will be formed at each stage ≥ 2 just as new sets are formed at each stage ≥ 1 on the iterative conception of sets. Moreover, with the right background assumptions—in particular, assumptions concerning how far the stages extend—we could even show that the axioms of ZF hold in the set-structures at some stage or other. At no stage will there be anything that could plausibly be called *the* set structure. And while the operation of abstraction that takes us from a system to its structure may not engender a relation of dependence on which set-structures at later stages in the hierarchy depend on those at previous stages, on the picture I'm considering, the structures at previous stages had to be available before the structures at later stages could be formed.

3.2 Dependence and potentialism

An increasingly popular position in the philosophy of mathematics is *potentialism*: the view that the universe of sets is inherently potential in character.[25] The core idea is that all that's required for the possible existence of a set is the co-existence of its members: once the members exist, the set *can* exist. So, whereas on the iterative conception of sets, set existence is a matter of co-existence at a stage, for the potentialist, (possible) set existence is a matter of (possible) co-existence simpliciter. One immediate consequence is that there could have been more sets than there actually are. For example, let xx be the non-self-membered sets. By Russell's paradox, the xxs *don't* form a set. But according to the potentialist, they *could*. So, there could have been a set of the xxs even though there isn't actually such a set.

As attractive as potentialism may be, it is unavailable to the orthodox non-eliminativist.[26] To see this, let s be the set structure. According to potentialism, there could have been a set that doesn't actually exist. For the orthodox non-eliminativist, that means there could have been a place x in s that doesn't actually exist. But s actually exists. So it must actually exist without x. Thus, x does not depend on s or on any sets that actually exist, contradicting the dependency claims of orthodox non-eliminativism. In general, if there could have been more sets than there actually are, the sets there actually are don't depend on at least some of the

[24] Consider, for example, the system of all set-structures under the relation $\in^*$ extended with a new object to act as its top element. It's easy to see that this new system is a set-system: it's well-founded, extensional, and has a top element. But because it's well-founded, it cannot itself determine a set-structure.

[25] See, for example, Linnebo [2010], Linnebo [2013], Studd [2013], Studd [2019], Parsons [1983], and Hellman [1989].

[26] Orthodox modal structuralism is, at least in spirit, already a form a potentialism. See [Roberts, 2019, p. 827-8] for discussion.

sets there could be. The sets there actually are exist perfectly well without them. Similarly, the set structure exists perfectly well without them.

In contrast, the sets as structures view is consistent with potentialism. Again, although a set-structure s cannot exist without its places and those places can't exist without each other, *other* set-structures can exist perfectly well without s. In particular, $s_\emptyset$ can exist without $s_{\{\emptyset\}}$. In general, it's consistent to assume that lots of set-structures are merely possible while granting the core dependency claim.

More fundamentally, potentialism gets transposed into the non-eliminative setting as another compelling response to the challenge I noted above: namely, to provide an account of the dividing line between the systems that determine structures and those that don't. According to the envisioned response, all that's required for the possible existence of a structure is the existence of a corresponding system: once the system exists, the structure *can* exist. On this view, possible structure existence would be matter of the possible existence of a system of which it's the abstract form. With the right background assumptions, we could show that the axioms of ZF hold in the potential set-structures.

3.3 Metaphysical entanglement

As I pointed out in section 2, Stability is a non-negotiable commitment for the orthodox modal structuralist, since it comprises the Putnam-translations of instances of the vacuous quantifier axiom schema. I also pointed out that unlike the vacuous quantifier axiom schema, Stability makes a substantial demand on modal space. It captures the cash value for Putnam-translation of the claim that each possible ZFC2 system has access to the very same sets (up to isomorphism). It tells us, in other words, that when it comes to Putnam-translation, modal space looks the same in all directions under end-extension. An example will help to illustrate. Let S be a possible ZFC2 system, and suppose that there could be a possible ZFC2 system S' in which there are infinitely many inaccessible cardinals. Thus, in S', the limit of the first countably-many inaccessible cardinals, κ_ω, exists. As the results in Roberts [2019] show, it follows that the Putnam-translation of the claim that κ_ω exists is true from the perspective of S', and thus true simpliciter. Formally:

$$(\kappa_\omega \text{ exists})^{pt}$$

Stability tells us that the Putnam-translations of sentences are true if, and only if, they are true from the perspective of absolutely all possible ZFC2 systems. So, in particular, it tells us that:

$$(\kappa_\omega \text{ exists})^{pt} \leftrightarrow \Box\forall S(\kappa_\omega \text{ exists})^{pt}_S$$

Thus:

$$(\kappa_\omega \text{ exists})^{pt}_S$$

It follows, again by the results in Roberts [2019], that S must have a possible end-extension containing infinitely many inaccessible cardinals.

Without knowing more about S, though, why think this is true? Its domain can comprise absolutely any objects whatsoever, as long as there are enough of them. And it certainly need not comprise so many objects that it contains infinitely many inaccessible cardinals. So, it's at least not obvious that modal space would oblige to supply the required end-extension. You might be tempted at this point to recall the quasi-categoricity of ZFC2, which says that any two ZFC2 systems are either isomorphic, or one is isomorphic to an inaccessible rank of the other. Applied to S and S', it would imply that either S' is isomorphic to an inaccessible rank of S or to S itself, or S is isomorphic to an inaccessible rank of S'.[27] In the first two cases, S already contains infinitely many inaccessible cardinals and so will trivially serve as the required end-extension. In the last case, the thought might go, we can surely use S' to construct the required end-extension of S. We just take a suitably large number of things from the domain of S', add them to S, and then extend S's relation so as to end up with an end-extension of S isomorphic to S'. In the modal setting, however, this is much too quick. The envisioned construction presupposes that S and S' can exist together. Otherwise, we couldn't freely add things from S' to S. Now, it turns out that we can indeed use this kind of construction to prove Stability *assuming* that any two possible ZFC2 systems can co-exist. More precisely, in MSST - Stability, Stability is provable from:[28]

(Compossible$_{\mathsf{ZFC2}}$) $$\Box\forall S\Box\forall S'\Diamond(ES \wedge ES')$$

But Compossible$_{\mathsf{ZFC2}}$ is highly contentious. Consider the following example from Williamson [2013]. We have a knife handle h and two knife blades b_1 and b_2. It's metaphysically possible for there to be a knife k_1 constructed from h and b_1 and also metaphysically possible for there to be a knife k_2 constructed from h and b_2. But it seems to be metaphysically impossible for k_1 and k_2 to exist together:[29] that is, k_1 and k_2 seem to be *incompossible*.[30]

[27] It's a subtle issue whether the quasi-categoricity theorem is even available in the modal setting. Whether it is depends on how exactly we're thinking about systems and isomorphisms. For example, in BT—where isomorphisms are understood as pluralities (see below)—the quasi-categoricity theorem is provable for any two co-existing ZFC2 systems, but not for any two arbitrary possible ZFC2 systems. Similarly, if we instead understand isomorphisms as second-order relations in the framework of higher-order logic, the examples in footnote 38 suggest that there may be possible ZFC2 systems which are definably isomorphic but for which there is no corresponding isomorphism. What the following points show is that even *granting* the quasi-categoricity theorem for any two possible ZFC2 systems, there is a further non-trivial step to obtain Stability.

[28] See [Roberts, 2019, p.831-2]. In modal logics weaker than S5, Compossible$_{\mathsf{ZFC2}}$ need not suffice to prove Stability. In particular, without the 5 axiom, it the possibilities in which S exists may not be accessible from the possibilities in which S' exists or vice versa. As a consequence—and despite its name—Compossible$_{\mathsf{ZFC2}}$ won't guarantee that S and S' can co-exist. (To see this, note that the model in the proof of theorem 5.40 in [Roberts, 2019, p. 856] can easily be modified to satisfy the axioms of S4 plus NNE together with its extension to systems—we just let $\langle\alpha, n\rangle$ access $\langle\beta, m\rangle$ when either $\alpha = n = 0$ or $n = m$ and $\alpha < \beta$—and thus satisfy Compossible$_{\mathsf{ZFC2}}$ but not Stability.)

It turns out that we can achieve the intended effect of Compossible$_{\mathsf{ZFC2}}$ using NNE—and its extension to systems—together with the G axiom of modal logic, which says that what's possibly necessary is necessarily possible (formally: $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$). In particular, over the modal logic we obtain by extending S4 with G and NNE—and its extension to systems—it turns out that MSST - Stability proves Stability, given new, but very natural, background assumptions. Indeed, in this modal logic, MSST - Stability proves φ^{pt} if and only if $\mathsf{Z}^* +$ In proves φ (and thus, by theorem 1, if and only if MSST proves φ^{pt} in S5). See claim 2 in the appendix for details.

This extension of S4 is the modal logic typically employed by non-structuralist potentialists (see, for example, Linnebo [2010], Linnebo [2013], Studd [2013], and Studd [2019]). But, as [Linnebo, 2013, p. 209] notes, for these potentialists G is motivated on the basis of the thought that any two domains of sets can be brought together into a further domain. And although that thought *is* plausible when we're dealing with genuine pure sets, it's precisely what's at issue when we're dealing with ordinary objects playing the role of sets.

[29] Assuming they have their parts essentially.

[30] Again, the necessitist denies this. For Williamson [2013], the seeming incompossibility of k_1 and k_2 is not due to an incompatibility in their existence, but rather to an incompatibility in their having certain fundamental properties. In particular, although k_1 and k_2 necessarily co-exist for Williamson, they cannot both be "chunky". Of course, there's room in logical space for a contingentist who agrees with the necessitist that any two possible things or systems can co-exist—explaining away purported cases of incompossibility as the necessitist does—whilst rejecting the (strong) necessitist claim that absolutely all possible objects and

Once we accept the possibility of incompossible objects, there seems to be no good reason why they couldn't populate the domains of ZFC2 systems in such a way as to make those systems themselves incompossible, thus invalidating $\mathsf{Compossible}_{\mathsf{ZFC2}}$. Indeed, as systems are understood in BT, this is almost immediate. In BT, the domain of a system is a plurality. And pluralities are typically taken to be nothing over and above the things they comprise.[31] In particular, they're taken to be *downward dependant* on the things they comprise: if a plurality comprises at least o, then it cannot exist without o.[32] It follows that if k_1 plays the role of the empty set in S and k_2 plays the role of the empty set in S', then given that k_1 and k_2 are incompossible, S and S' will be incompossible.[33] Call this the *problem of incompossibles*.[34]

One response to the problem of incompossibles is to weaken $\mathsf{Compossible}_{\mathsf{ZFC2}}$ to the claim that any possible ZFC2 system can co-exist with an isomorphic copy of any other. Formally:

$$(\mathsf{Compossible}^{-}_{\mathsf{ZFC2}}) \qquad \Box\forall S\Box\forall S'\Diamond\exists S''(ES \wedge S \cong S'' \wedge \Diamond(ES'', S'))$$

where $S \cong S'$ abbreviates the claim that there is an isomorphism between S to S'. Given this principle, we could use S'' instead of S' to build the relevant end-extension of S. And since S'' need not be incompossible with S—even if S' is incompossible with S—it is perfectly consistent with failures of $\mathsf{Compossible}_{\mathsf{ZFC2}}$. This new principle, however, falls foul of other

systems can co-exist all at once. Indeed, I'll suggest below that on one way of thinking about the modal structuralist's notion of possibility, that view might be plausible.

[31]See Roberts [2022] for discussion.

[32]In section 4.2, downward dependence is formalised as Plural Stability.

[33]A natural alternative is to understand systems in the framework of higher-order logic as consisting of a (second-order) property to act as its domain together with a (second-order) relation to act as its relation. Although properties are not in general downward dependent on the things they apply to, it's a substantial open metaphysical question whether some may be in such a way as to lead to similar failures of $\mathsf{Compossible}_{\mathsf{ZFC2}}$. For example, it's hard to see how one might individuate a mereological atom in circumstances where it fails to exist. Williamson [2013] has argued that in such circumstances the haecceity of the mereological atom must also fail to exist. Haecceities of mereological atoms, in other words, do seem to be downward dependent on the things they apply to. The thought generalises: the property of being one of some mereological atoms would seem to be downward dependent on those atoms. Furthermore, Fritz and Goodman [2017] have argued that there could be incompossible objects o and o' for which neither can be individuated when they fail to exist. Following Williamson [2013] again, their haecceities would then be downward dependent on them and thus themselves incompossible. The thought generalises: the property of being one of some mereological atoms together with o would seem to be downward dependent on those atoms and o, and similarly for o'. It would then follow that those properties themselves are incompossible, and thus so too are any possible ZFC2 systems that have them as domains. This also suggests a case in which there might fail to be a (second-order) relation coding an isomorphism between structures, despite their being definably isomorphic. We simply let S and S' have domains as described, with o playing the role of the empty set in S and o' playing the role of the empty set in S'. Any (second-order) relation that acts as an isomorphism between them would uniquely associate o with o'. But it would seem that no such relation could be individuated when o and o' fail to co-exist. Thus, given that o and o' will always fail to co-exist, no such relation could exist.

[34]One might be tempted at this point to invoke so-called *outer* quantifiers. In terms of Kripke models: the outer quantifier at a world w ranges over all the objects and systems in the domains of worlds accessible from w, whereas the inner quantifier only ranges over the objects and systems in the domain of w. Although possible objects and systems might be incompossible with respect to the inner quantifiers, they will, almost by definition, be compossible with respect to the outer quantifiers. Formulated in terms of the outer quantifiers, then, $\mathsf{Compossible}_{\mathsf{ZFC2}}$ will be something of a triviality. Thus, the thought might go, Stability could then be proved from such a version of $\mathsf{Compossible}_{\mathsf{ZFC2}}$ along the lines mentioned above. This would be a mistake. That proof idea requires S and S' to co-exist in such a way that a suitable end-extension of S can be constructed using the elements of S'. But co-existence in the outer sense certainly does not guarantee that. For example, if domains are pluralities and everything in the domain of S is incompossible with everything in the domain S', then no such end-extension can exist in either the inner or outer sense.

potential metaphysical constraints. For example, Linnebo [2018b] has raised the possibility of *metaphysically shy* objects. As he puts it, these are objects:

> which can live comfortably in universes of small infinite cardinalities, but which would rather go out of existence than to cohabit with a larger infinite number of objects" [Linnebo, 2018b, p.257].

If S contains a metaphysically shy object that cannot co-exist with more objects than there are in S, then why think that S can co-exist with any isomorphic copy of any larger system? Again, assuming domains are pluralities, it's immediate that it cannot.[35]

The possibility of metaphysically shy objects doesn't merely cast doubt on $\mathsf{Compossible}^{-}_{\mathsf{ZFC2}}$, it directly targets $\mathsf{Stability}$ itself. As I pointed out at the start of this section, if S contains no inaccessible cardinals, but there are possible ZFC2 systems containing infinitely many inaccessible cardinals, then $\mathsf{Stability}$ implies that S must have a possible ZFC2 end-extension containing inaccessibly many more objects than there are in S. But when S contains a metaphysically shy object that cannot co-exist with more objects than there are in S, why think such end-extensions are possible? Indeed, assuming domains are pluralities, it's immediate that they aren't.[36] Call this the *problem of metaphysical shyness*.[37]

The next natural step would be to weaken $\mathsf{Compossible}_{\mathsf{ZFC2}}$ further to the claim that isomorphic copies of any two ZFC2 systems can co-exist. To claim, in other words, that any two ZFC2 systems are *co-realisable*. Formally:

$$(\mathsf{Co\text{-}Realisability}_{\mathsf{ZFC2}}) \quad \Box\forall S\Box\forall S'\Diamond\exists S'', S'''(\Diamond(ES'', S \wedge S \cong S'') \wedge \Diamond(ES''', S' \wedge S' \cong S'''))$$

Since S'' and S''' need not contain incompossibles nor metaphysically shy objects—even when S and S' do—this principle seems to side steps both the problem of incompossibles and the problem of metaphysical shyness. Unfortunately, unlike $\mathsf{Compossible}_{\mathsf{ZFC2}}$ and $\mathsf{Compossible}^{-}_{\mathsf{ZFC2}}$, $\mathsf{Co\text{-}Realisability}_{\mathsf{ZFC2}}$ fails to imply $\mathsf{Stability}$ in MSST - $\mathsf{Stability}$.[38]

None of this is fatal to orthodox modal structuralism, of course. It simply shows is that, on the face of it, the view is entangled with a number of non-trivial metaphysical issues. Absent some compelling response to those issues, that's a substantial cost. In contrast, the sets as structures view avoids them. As I'll show in section 4.2, its success merely depends on increasing levels of co-realisability for set-systems. In particular, over BT, the co-realisability of any two set-systems suffices to prove the ^{tr}-translations of all the theorems of ZF minus Pairing, Infinity, Powerset, and Collection; and the disjoint co-realisability of any collection of set-systems no more numerous than the subpluralities of any given plurality suffices to

[35] Again, if we think of domains as properties, then in principle S may co-exist with a larger system S' as long as S''s domain is not downward dependent on the things it applies to. The discussion in footnote 34, however, suggests that such co-existence is in no way guaranteed.

[36] If domains are properties, then in principle S may have such an end-extension. But it's far from obvious that such end-extensions will in general be possible, since it's unclear how they could be individuated, given that S itself contains a metaphysically shy object.

[37] In addition to individually metaphysically shy objects, there might also be *collectively* metaphysically shy objects. That is, some objects, each of which could live comfortably in universes of arbitrary size, but which would collectively rather go out of existence than co-exist together with any more objects. Even if we rule out, or ignore, individually metaphysically shy objects, collectively metaphysically shy objects would cause similar problems.

[38] See theorem 5.40 and remark 5.41 in Roberts [2019].

further prove the ^{tr}-translations of Pairing, Powerset, and Collection.

There are two more things I want to do before I move on. First, I want to outline two natural responses the modal structuralist might make to the preceding discussion. Second, I want to outline and respond to two other issues Linnebo [2018b] raises for orthodox modal structuralism that seem to equally target the sets as structures view.

The problems of incompossibles and metaphysical shyness seem to assume that the modal structuralist is working with a notion of metaphysical possibility. But that assumption is not mandatory. Each notion of possibility gives rise to an interpretation of MSST. Some will render its axioms false, of course, whereas some (hopefully) don't. And, in fact, a number of modal structuralists have opted to work with a notion of possibility broader than metaphysical possibility: namely, logical possibility.[39] On that interpretation, the above worries seem to lose their force.[40] Perhaps there could be metaphysically incompossible objects, but it's unclear that there could be *logically* incompossible objects. In particular, given any two logically possible ZFC2 systems, there doesn't appear to be any logical barrier to subsuming them in a single possibility, no matter what populates their domains. Similarly, perhaps there could be metaphysically shy objects, but it's unclear that there could be *logically* shy objects. For any logically possible object, there doesn't appear to be any logical barrier to subsuming it in a single possibility with as many objects as there are in the domain of any possible ZFC2 system. Of course, there will have to be *some* limits to logical compossibility. Assuming that necessarily, no matter what objects there are, there could have been more, it must be impossible for absolutely all logically possible objects to co-exist.[41] The envisioned view thus raises deep questions about the limits of logical compossibility. What *is* the dividing line between those collections of logically possible objects that are logically compossible and those that aren't? On the envisioned view, orthodox modal structuralism might best be seen as articulating general constraints on the answer to this question.[42,43]

Another natural response to the problems of incompossibles and metaphysical shyness is to change the Putnam-translation scheme. Strictly speaking, that would require a rejection

[39]See, for example, Hellman [1989], Hellman [1990], and Berry [2022]. Indeed, many non-structuralist potentialists also seem to appeal to a notion of logical possibility. In particular, they work with a notion of *interpretational* possibility, according to which for it to be possible that φ is for it to be possible to reinterpret the language in a suitably controlled way so that φ is true (see Linnebo [2018a], Linnebo [2018b], and Studd [2019]). As Linnebo points out:

> The most well-known example of an interpretational modality is the one used in the standard Tarskian definition of logical consequence. [Linnebo, 2018b, p. 264]

And it's natural to identify logical possibility with the *broadest* interpretational possibility (keeping logical expressions fixed).

[40]Though see the discussion in section 4.3 for some other worries.

[41]See [Roberts, 2019, p. 831-832] for discussion.

[42]It's worth noting that as long as the notion of logical possibility is in good standing and there logically could be ZFC2 systems or set-systems beyond those there metaphysically could be, then without some reason to the contrary, there is pressure on the orthodox modal structuralist to include such systems in the scope of their view, and thus to work with such a notion. Failing to do so would risk another arbitrary and unjustified curtailing of Cantor's paradise.

[43]Another option would be to work with systems comprising objects that are generally metaphysically compossible. Mereological atoms, for example, might fit the bill here. Of course, this would involve substantial metaphysical assumptions about the possibility of enough such objects to do the required work.

of orthodox modal structuralism as I've defined it. But there are modifications of Putnam-translation that avoid those problems whilst retaining its spirit. In particular, we could follow Berry [2022] and work with end-extensions *up to isomorphism*, rather than end-extension simpliciter. To see the idea, consider again the claim that every set is a member of some set. According to the proposal, we would translate that as the claim that necessarily for any ZFC2 system S and any object o in its domain, there could be a ZFC2 system S' and an isomorphism i between S and S' such that there could be a ZFC2 system S'' *end-extending* S' and an object o' in S'''s domain, such that according to S', $i(o)$ is a member of o'. In general, the modified translation schema could defined by recursion as follows.

- $(x \in y)_S^{pt^*} = S \models x \in y$
- $(x = y)_S^{pt^*} = x = y$
- ${}^{pt^*}$ commutes with the connectives
- $(\exists x \varphi(x, \vec{y}))_S^{pt^*} = \Diamond \exists S', i, \vec{z}(ES \wedge i : S \cong S' \wedge i(\vec{y}) = \vec{z} \wedge \Diamond \exists S'' \sqsupseteq S' \exists x \in S'' \varphi(x, \vec{z})_{S''}^{pt^*})$
- When φ is a sentence, we let φ^{pt^*} abbreviate $\varphi_\emptyset^{pt^*}$

And although this involves a further jump in complexity beyond Putnam-translation, it turns out that its success merely depends on Co-Realisability$_{\mathsf{ZFC2}}$. More precisely, let Stability be reformulated in terms of ${}^{pt^*}$-translation, and let Extendibility be reformulated in terms of end-extension up to isomorphism. Then I claim we have the following analogue of theorem 3.

Claim 1. MSST - Stability + Co-Realisability$_{\mathsf{ZFC2}}$ *proves* φ^{pt^*} *if and only if* Z^* + In *proves* φ.

It follows immediately from claim 1 that MSST - Stability + Co-Realisability$_{\mathsf{ZFC2}}$ proves Stability. I omit the proof of claim 1 here, but it's not hard to reconstruct it on the basis of the proof of theorem 2.4 in Roberts [2019] and the proof of theorem 5 in the appendix.

Linnebo [2018b] raises two further issues for the orthodox modal structuralist, concerning applied, or impure, set theory. Unlike the problems of incompossibles and metaphysical shyness, these issues seem to arise equally for the sets as structures view. They depend on a natural thought about how to extend the sets as structures view to accommodate impure sets. The idea is that, first, we change our definition of set-system so that Extensionality only holds for non-terminal nodes and designate one terminal node to play the role of the empty set. Call such systems *impure-set-systems*. A possible object o is then taken to be a member of an impure-set-system precisely when it's a terminal node immediately below the top element and distinct from the empty set element. On this approach, sets are played by systems, whereas non-sets are played by themselves.

The first issue Linnebo raises concerns impure set theory over metaphysically shy objects. Impure set theory tells us that every non-set is a member of some infinite set. The truth of that claim in the impure-set-systems implies that every possible object could be a terminal node of some infinite impure-set-system. But given that there might be metaphysically shy objects that cannot exist in infinite universes, why think that this is so? Indeed, assuming that the domains of impure-set-systems are pluralities, it provably isn't. The second issue Linnebo raises concerns the conditions under which impure-set-systems might be available. To see the problem, suppose we want to apply impure set theory to the physical world. Suppose, for example, that we want to do set theory over the actual electrons under the relation of entanglement. Then it might be that the only possibilities in which there are, say, infinite impure-set-systems containing all of the electrons are possibilities in which the relation of entanglement is different from what it actually is. Anything we then come to conclude using those impure-set-systems will not be guaranteed to be applicable to the actual electrons and their actual relation of entanglement. In general, it might be that we can only find suitable impure-set-systems in possibilities where the non-set-theoretic facts are different from what they actually are. Anything we then come to conclude using those impure-set-systems will not be guaranteed to apply to the actual non-set-theoretic world. Call this the *problem of control*.

Luckily for the sets as structures view, there are other ways to accommodate impure sets. Here's one approach. The key idea is that just as set-systems comprising arbitrary objects play the role of sets, arbitrary objects should play the role of non-sets. Let me explain. For simplicity, I'll make free use of possible worlds talk. We start in a world w containing some objects together with some relation on them. For concreteness, we can assume that the objects are the electrons and the relation is entanglement. Let S be the corresponding system.[44] Suppose now that S is co-realisable with a set-system containing a pair of sets isomorphic to S.[45] So, let w' be a world containing an isomorphic copy S' of S, and let w'' be a world containing a set-system S'' whose members x and y play the role of a set and a set of ordered pairs, respectively, that are isomorphic to S'. Now, let φ be a claim about electrons and entanglement. It will be true in w that φ is equivalent to φ^S; it will be true in w' that φ^S is equivalent to $\varphi^{S'}$; and it will be true in w'' that $\varphi^{S'}$ is equivalent to $(\varphi^{\langle x,y\rangle^{S''}})^{tr}$ (where $\varphi^{\langle x,y\rangle^{S''}}$ is the claim that φ is true in x and y according to S'').[46] The proposal is then that given such a coding of S as x and y in the set-system S'', we do our pure set theory as usual over S'', inferring, for example, that $(\varphi^{\langle x,y\rangle^{S''}})^{tr}$. Then by the mentioned equivalences, we first conclude that $\varphi^{S'}$ holds in w'', then that $\varphi^{S'}$—and thus φ^S—holds in w', and finally that φ^S—and thus φ—holds in w.[47] And this reasoning is available even if the only worlds where S' and S'' exist are worlds where $\neg\varphi$.

3.4 Face value semantics

In section 2, I mentioned that one of the main benefits of non-eliminativism over eliminativism is that it allows for a face value semantics. What appear to be ordinary first-order quantifiers over a non-potential domain of abstract objects in language of set theory *really are* such according to the non-eliminativist. It's just that those objects are the places in the set structure. Since the non-eliminativist sets as structures ^{t}-translation that I outlined in section 2 takes the language of set theory to concern a domain of abstract objects—the set-structures—it similarly provides a face value semantics. The same cannot be said for the eliminativist sets as structures ^{tr}-translation. That translation takes first-order quantifiers in the language of set theory to range a potential domain of set-systems, and it takes identity

[44]The existence of S is guaranteed in BT, but see the discussion in section 4.3 for some relevant worries here.

[45]The possible existence of such a system would follow, for example, from the claim that (i) S's domain could be well-ordered—and therefore be the domain of a set-system S'—and (ii) that S' could be a member of a set-system whose domain is closed under (what it thinks is) set-theoretic pairing.

[46]A simple induction on the complexity of φ would suffice to establish this equivalence in BT.

[47]Doing so, we use the fact, established in BT, that satisfaction for systems is stable across worlds where those systems exist. See Lemma 1 and the comments immediately after.

to be possible isomorphism between them. Nevertheless, ^{tr}-translation is much *closer* to a face value semantics than Putnam-translation. For according to ^{tr}-translation, set theoretic quantifiers really do range over a domain of entities—its just that the domain is potential in character and the entities are systems. Similarly, under ^{tr}-translation, membership and identity really are relations on those entities. Furthermore, the intuitive constraint that φ^{tr} imposes on the world is much closer to the constraint imposed by the face value reading of φ than that imposed by φ^{pt}. For there to be an empty set according to ^{tr}-translation is just for it to be possible that at least one thing exists. Similarly, for there to be an infinite set is just for it to be possible that there are at least infinitely many things. According to Putnam-translation, both claims require it to be possible for there to exist an inaccessible infinity of things.[48]

4 A formal theory for eliminative sets as structures

In this section, I'll provide a formal theory for the eliminative sets as structures view and show how it can be used to provide an interpretation of set theory via ^{tr}-translation. Before I do that, though, I'll start with a warm up in ZF.

4.1 Sets as systems in ZF

In ZF, the most natural way to think about a system is as a pair of sets: one set to play role of the domain, and one to play the role of the relation.

Definition 1. *A* **system** *is simply two sets x and y. We say that y relates w to z—or that w is below z—precisely when $\langle w, z\rangle \in y$.*

We then have the following obvious definitions of satisfaction and set-system.

Definition 2. *Let φ be a formula in the language of monadic second-order set theory, and let $\varphi^{x,y}$ be the result of replacing occurrences of $\exists w$ in φ with $\exists w \in x$, occurrences of $\exists X$ with $\exists z \subseteq x$, and occurrences of $w \in z$ with $\langle w, z\rangle \in y$ (making sure to avoid clashes of variables). A system x, y* **satisfies** *φ precisely when $\varphi^{x,y}$. I'll sometimes write $\varphi^{x,y}$ as $x, y \vDash \varphi$.*

Definition 3. *A system is a* **set-system** *when it satisfies* $\mathsf{Extensionality}$, $\mathsf{Foundation}_2$, *and* Top. *Let $S, S', S'', \ldots$ etc range over set-systems and let $\mathsf{top}(S)$ denote the unique top element of S. If S is x, y, I'll say that w is in S, or contained in S, or a member of S, or $w \in S$, when $w \in x$. Let $x < \mathsf{top}(S)$ abbreviate the claim that x is below the top element of S: formally, $x \in S \wedge \langle x, \mathsf{top}(S)\rangle \in y$.*

Next, we have analogues for set-systems of identity and membership for sets.

[48] Even if we replace $\mathsf{ZFC2}$ systems by R systems in Putnam-translation—as, among other things, Berry [2022] does—the point stands. For the singleton of a set-system S to exist according to ^{tr}-translation is just for it to be possible that there be a set-system extending S (up to isomorphism) with one further object. According to Putnam-translation modified for R systems, the analogue of that claim requires that there could be as many objects as the subpluralities of S's domain. We could avoid this by working instead with the weakest kind of systems of sets: namely, the well-founded extensional systems (see section 2.7.4 of Roberts [2019]). If we also work with end-extension up to isomorphism (as in the previous section), then the resulting view will be a perverse variant of the sets as structures view. This shows, I think, that the sets as structures view is precisely what you get when you strip orthodox modal structuralism of its superfluous assumptions.

Definition 4. *Let $S =^* S'$ abbreviate the claim that S and S' are isomorphic, and let $S \in^* S'$ abbreviate the claim that there is some $x < \mathsf{top}(S')$ for which $S =^* S' \restriction x$ (where $S' \restriction x$ is the system you get by restricting S' to x and the objects transitively below x in S').*

We can now define a translation from the language of set theory, $\mathcal{L}_\in$, to itself which re-interprets claims about sets as the corresponding claims about set-systems.

- $(x = y)^* = S =^* S'$
- $(x \in y)^* = S \in^* S'$
- * commutes with the connectives.
- $(\exists x \varphi)^* = \exists S \varphi^*$

Then we have the following folklore result which tells us that claims about the sets are equivalent to the corresponding claims about set-systems.

Theorem 4 (ZF)**.** *Let $\varphi \in \mathcal{L}_\in$ be a sentence. Then:*

$$\varphi \quad \leftrightarrow \quad \varphi^*$$

I'll now transpose all of this into the modal structural setting.

4.2 Sets as possible systems

To remain nominalistically kosher, modal structuralists can't think of systems as pairs of sets. So, how exactly should they think of systems? For Hellman [1996], a system comprises *some* objects as a domain together with *some* objects—which, due to their mereological structure, behave suitably like ordered pairs—as a relation. Systems, in other words, are pairs of pluralities for Hellman. I'm going to follow Hellman in this, but I'll depart from him in taking as primitive the notion of a relation-as-plurality, rather than understand it mereologically. To do this, I'll make use of a new three-place predicate '$\langle x, y\rangle \prec xx$'—intended to express the claim that x and y are related (in that order) by the xxs—governed by some simple but general axioms. As we'll see in section 4.3, with suitable adjustments, my theory will then admit of many different interpretations, only one of which is mereological. My hope is that modal structuralists can therefore make use of my results, whatever their favourite way of thinking about systems.

I will work in the language of plural modal logic extended with '$\langle x, y\rangle \prec xx$', which I'll call $\mathcal{L}$. The modal logic will be a positive free version of S5.[49] To that, we first add some general background assumptions concerning pluralities and relations-as-pluralities. The first three follow from the nothing over and above conception of pluralities.[50] In terms of domains, they tell us that there is always a domain of all and only the φs; that domains don't change the things they comprise across possibilities where they exist; and that domains comprising the same things are identical.

(Plural Comp) $\qquad \exists xx \forall x (x \prec xx \leftrightarrow \varphi)$

[49] See, for example, the axiomatisation in Roberts [2019].

[50] Indeed, Roberts [2022] articulates a sense in which these principles capture that conception within the language of plural modal logic.

(Plural Stability) $x \prec xx \to \Box(Exx \to Ex \wedge x \prec xx)$

(Plural Extensionality) $\Box\forall x[\Diamond(x \prec xx) \leftrightarrow \Diamond(x \prec yy)] \to xx = yy$

where 'Ex' abbreviates '$\exists y(y = x)$' and 'Exx' abbreviates '$\exists yy(yy = xx)$'. Next, we have a relational version of Plural Comp, which says that there is always a relation-as-plurality relating all and only the x and y (in that order) for which $\varphi(x, y)$. Formally:

(Relation Comp) $\exists xx\forall x, y(\langle x, y\rangle \prec xx \leftrightarrow \varphi)$

Finally, we have a relational version of Plural Stability, which says that relations-as-pluralities don't change the things they relate across possibilities where they and those things exist. Formally:[51]

(Relation Stability) $\langle x, y\rangle \prec xx \to \Box(Exx, x, y \to \langle x, y\rangle \prec xx)$

Let BT denote the theory in $\mathcal{L}$ comprising Plural Comp, Plural Stability, Plural Extensionality, Relation Comp, and Relation Stability (together with the results of prefixing them with any string of quantifiers and $\Box$s) over the (positive free S5) modal logic.

Definitions 1 to 3 have obvious analogues in $\mathcal{L}$.

Definition 5. *A* **system** *is simply two pluralities xx and yy. We say that yy relates x to y—or that x is below y—precisely when $\langle x, y\rangle \prec yy$.*

Definition 6. *Let φ be a formula in the language of monadic second-order set theory. Let $\varphi^{xx,yy}$ be the result of replacing occurrences of $\exists x$ in φ with $\exists x \prec xx$, occurrences of $\exists X$ with $\exists zz \subseteq xx$, and occurrences of $x \in y$ with $\langle x, y\rangle \prec yy$. A system xx, yy* **satisfies** *φ precisely when $\varphi^{xx,yy}$. I'll sometimes write $\varphi^{xx,yy}$ as $xx, yy \vDash \varphi$.*

Definition 7. *A system is a* **set-system** *when it satisfies* Extensionality, Foundation$_2$*, and* Top*. Let $S, S', S'', \ldots$ etc range over set-systems and let* $\mathsf{top}(S)$ *denote the unique top element of S. If S is xx, yy, I'll say that x is in S, or contained in S, or a member of S, or $x \in S$, when $x \prec xx$. Let $x <$* $\mathsf{top}(S)$ *abbreviate the claim that x is below the top element of S: formally, $x \in S \wedge \langle x, \mathsf{top}(S)\rangle \prec yy$.*

The analogues of identity and membership are more complicated than they were in ZF. As we saw in section 3.3, systems containing incompossible objects will themselves be incompossible. But in $\mathcal{L}$, the most natural way to think about isomorphisms is as structure preserving relations-as-pluralities, and BT gives us no guarantee that such pluralities exist for incompossible set-systems, even when those set-systems are intuitively isomorphic.[52] Indeed, if

[51]Note that this is slightly weaker than axiom (P3) in Roberts [2019], i.e.: $\langle x, y\rangle \prec xx \to \Box(Exx \to \langle x, y\rangle \prec xx)$. This makes the theory presented here a little more general than that in Roberts [2019]. But it doesn't affect the main results of Roberts [2019]. To see this, note that the analogue of (P3) for systems is provable from Plural Stability and Relation Stability. That is, Plural Stability and Relation Stability imply: $x, y \prec xx \wedge \langle x, y\rangle \prec yy \to \Box(Exx, yy \to \langle x, y\rangle \prec yy)$. And it's that analogue of (P3) that the results in Roberts [2019] rely on (beyond the other principles presented here).

[52]For example, when S and S' are incompossible but differ only in their top elements.

relations-as-pluralities are downward dependent on the things they relate,[53] there provably can be no such isomorphisms. I therefore propose that we think of isomorphisms between possible set-systems in terms of isomorphisms-as-pluralities between their isomorphic copies, in the following way.

Definition 8. *Say that S and S' are* **isomorphic**, $S \cong S'$, *when there are some things xx coding an isomorphic relation between them. Say that S' is an* **isomorphic copy** *(or, simply, a* **copy***) of S, $S \cong^{\Diamond} S'$, when S' could have been isomorphic to S. Formally, $S \cong^{\Diamond} S'$ abbreviates:*

$$\Diamond(ES, S' \wedge S \cong S')$$

Say that S and S' are **identical***, $S =^* S'$, *when there could be isomorphic copies of S and S' that are themselves isomorphic. Formally, $S =^* S'$ abbreviates:*

$$\Diamond\exists S'', S'''(S'' \cong S''' \wedge S'' \cong^{\Diamond} S \wedge S''' \cong^{\Diamond} S')$$

Finally, say that S and S' are **co-realisable** *when isomorphic copies of S and S' can co-exist. Formally, S and S' are co-realisable when:*

$$\Diamond\exists S'', S'''(S'' \cong^{\Diamond} S \wedge S''' \cong^{\Diamond} S')$$

Identity* will be our analogue of identity, and the analogue of membership is then immediate.

Definition 9. *Say that S is a* **member*** *of S', $S \in^* S'$, when it is possible that S' exists and $S =^* S' \upharpoonright x$, for some $x < \mathsf{top}(S')$ (where $S' \upharpoonright x$ is the system we get by restricting S' to x and the objects transitively below x in S').*

Given definitions 5 to 9, we have the promised translation schema from the language of set theory to $\mathcal{L}$.

- $(x = y)^{tr} = S =^* S'$
- $(x \in y)^{tr} = S \in^* S'$
- ^{tr} commutes with the connectives
- $(\exists x\varphi)^{tr} = \Diamond\exists S\varphi^{tr}$

It's not hard to see that for identity* to function properly, intuitively isomorphic set-systems will have to be co-realisable. The first substantial principle I'm going to adopt extends this to all set-systems. It says that any two set-systems are co-realisable.

(Co-Realisability) $\quad \Box\forall S\Box\forall S'\Diamond\exists S'', S'''(S'' \cong^{\Diamond} S \wedge S''' \cong^{\Diamond} S')$

It turns out that this already suffices to prove the ^{tr}-translations of a good chunk of set theory. First, it ensures that classical first-order logic is preserved under ^{tr}-translation. So, in particular, it implies that identity* is a congruence relation for ^{tr}-translation. Second,

[53]Formally:

$$\langle x, y\rangle \prec xx \rightarrow \Box(Exx \rightarrow Ex, y)$$

This might be the case if we take a mereological approach to relations-as-pluralities, and it *will* be the case if we think of relations as higher-level pluralities as in section 4.3.

it ensures that the ^{tr}-translations of the axioms of Extensionality, Foundation, Union, and Separation come out true.[54] To obtain the ^{tr}-translations of the remaining axioms of ZF—namely, the axioms of Pairing, Powerset, and Collection—I'm going to adopt a principle based on a natural strengthening of Co-Realisability. It will be helpful to approach this principle in a couple of steps.

We first strengthen Co-Realisability to the claim that any two possible set-systems are *disjointly* co-realisable.[55] This allows us to obtain a pair set-system for any two possible set-systems. To see this, suppose that S_0 and S_1 are disjointly co-realised by S_0' and S_1', and that S_0' and S_1' co-exist together with some object x which is in neither system. We can then build a set-system corresponding to the pair of S_0 and S_1 as follows. We start with S_0'. We take all of the elements of S_1' that don't correspond to any element of S_0',[56] and add them to S_0', extending its relation accordingly.[57] Finally, we add x as a top element to the new system to obtain the required pair set-system. The argument generalises to any collection of set-systems. For example, suppose that $S_0, S_1, ..., S_i, ...$ are disjointly co-realised by $S_0', S_1', ..., S_i', ...$, and that $S_0', S_1', ..., S_i', ...$ co-exist together with some object x which is an element of none of them. We can build a set-system corresponding to the set of $S_0, S_1, ..., S_i, ...$ as follows. We start with S_0'. We take all of the elements of S_1' that don't correspond to any element of S_0' and add them to S_0', extending its relation accordingly. We then do the same for S_2', and S_3', and so on. Finally, we add x as a top element to the new system to obtain a set-system corresponding to the set of $S_0, S_1, ..., S_i$.[58] Since the ^{tr}-translations of the axioms of Pairing, Powerset, and Collection tell us that various collections of possible set-systems are members* of some possible set-system, this means that we can obtain them from a suitably general disjoint co-realisability claim. The following, for example, would do the trick: any possible set-systems no more numerous than the subpluralities of a given plurality are disjointly co-realisable.

Instead of adopting such a claim, I will instead adopt its consequence for possible set-system membership*: namely, that any possible set-systems no more numerous than the

[54] In fact, as you'll see from the proof of theorem 7, a number of these can be established in BT alone.

[55] Formally:

$$\Box\forall S\Box\forall S'\Diamond\exists S'', S'''(S'' \cong^{\Diamond} S \wedge S''' \cong^{\Diamond} S' \wedge \neg\exists x \in S''(x \in S'''))$$

[56] Formally, these will be the $y \in S_1'$ for which $S_1' \restriction y$ is not isomorphic to $S_0' \restriction z$, for any $z \in S_0'$.

[57] Formally, let xx be the resulting domain. Then, we relate $x, y \in S_0'$ if they are related by S_0', we relate $x, y \in S_1' \cap xx$ if they are related by S_1', and we relate $x \in S_0'$ and $y \in S_1' \cap xx$ when $S_0' \restriction x$ is isomorphic to $S_1' \restriction z$ for some z below y in S_1'.

[58] The reasoning here implicitly assumes that we have an order on the S_i' along which we can carry out the construction. So, if we wanted to turn this into an explicit proof, we would need to assume something like a global well-ordering principle, according to which some relation-as-plurality always well-orders the objects. Since the domains of $S_0', S_1', ..., S_i', ...$ are disjoint, any such well-order would induce a well-order on those set-systems themselves. An alternative, though more drastic, way of making the reasoning explicit would be to work with a notion of set-system where identity is treated internally. Let me explain. Say that a system is a *set-system** if it's a triple of pluralities xx, yy, zz—where xx is taken to be the domain, yy its membership relation, and zz its identity relation—satisfying Extensionality, Foundation$_2$, and Top together with the first-order identity axioms. Then, instead of working with isomorphisms between set-systems, we would work with many-many relations between set-systems* that preserve their membership and identity relations, together with the resulting versions of identity* and membership*. The crucial lemma—which is provable in BT—is: for any system xx, yy satisfying Foundation$_2$ and Top, there is a unique (up to extension) identity relation zz on xx such that xx, yy, zz is a set-system*. It follows that if we simply take the union all the disjoint set-systems $S_0', S_1', ..., S_i'...$ with x 'on top', then (i) by the lemma, that determines a unique (up to extension) set-system* which (ii) contains (in the new sense of membership) each of them. It is not hard to see that, suitably adjusted, theorem 5 can be established for set-systems*.

subpluralities of a given plurality are the members* of some possible set-system.[59] Formally:[60]

(Collection) $\forall yy[\forall xx \subseteq yy \Diamond \exists S \varphi(xx, S) \to \Diamond \exists S' \Box \forall xx \subseteq yy \Diamond \exists S(\varphi(xx, S) \land S \in^* S')]$

Finally, since there's no getting around it, I will also adopt an axiom of infinity, which says that there could be a set-system corresponding to the first infinite ordinal. Formally:

(Infinity)
$$\Diamond \exists S(\exists x(x < \mathsf{top}(S))$$
$$\land \forall x, y \in S(S \vDash x \in y \lor x = y \lor y \in x)$$
$$\land \forall x < \mathsf{top}(S) \exists y < \mathsf{top}(S)(S \vDash x \in y))$$

Let SAS—for *Sets as Structures*—be BT + Co-Realisability + Collection + Infinity (together with the result of prefixing instances of Collection with any string of quantifiers and □s). Then, we can show that SAS proves the ^{tr}-translations of precisely the theorems of ZF.[61]

Theorem 5. *Let $\varphi \in \mathcal{L}_\in$ be a sentence. Then,* SAS *proves φ^{tr} just in case* ZF *proves φ.*

We now have two modal structural views in front of us: orthodox modal structuralism—as embodied in MSST—and the sets as structures view—as embodied in SAS. In section 3, I argued that the sets as structures view has a number of benefits over the orthodoxy. I'll finish this section by showing that, once the orthodox modal structuralist adds two further natural principles to MSST, the two approaches are equivalent. The orthodox modal structuralist has nothing to lose, and everything to gain, by switching to the sets as structures view.

In section 2, I worried that by ignoring merely possible ZFC2 systems, the eliminativist might unjustifiably curtail Cantor's paradise. A similar worry arises for the orthodox modal structuralist regarding set-systems. In particular, for all MSST tells us, there could be set-systems that aren't represented by elements of any possible ZFC2 system. But if such set-systems are possible, on what basis can the orthodox modal structuralist legitimately ignore them? What, as it were, is so special about the sets represented by elements of possible ZFC2 systems, rather than the sets represented by arbitrary possible set-systems? What if it turned out that the ^{tr}-translation of the generalised continuum hypothesis is false, but this is isn't witnessed in any possible ZFC2 system? Or, what if there could be measurable^{tr}, or Woodin^{tr}, or whatever, cardinals^{tr}, but not in any possible ZFC2 system?[62] In general, to ignore some possible set-systems seems to involve an arbitrary and unjustified curtailing of Cantors paradise. Arguably, then, the orthodox modal structuralist should adopt the claim

[59] As in the previous footnote, we could explicitly prove this from the disjoint co-realisability claim in a couple of ways (using the Infinity principle below to insure that we always have an extra object to serve as a top element).

[60] Collection is somewhat similar to the axiom of *Amalgamation* in [Berry, 2022, p. 100-101]. It is also analogous to the following strengthening of the axiom of Collection.

$$\forall x \subseteq y \exists z \varphi(x, z) \to \exists w \forall x \subseteq y \exists z \in w \varphi(x, z)$$

[61] If we add a global well-ordering principle to SAS, theorem 5 extends to ZFC.

[62] The converse worry for the sets as structures view—that it might miss some set represented in a possible ZFC2 system—doesn't arise, since every possible ZFC2 system is a top element away from being a set-system.

that for any possible set-system S, any possible ZFC2 system can be end-extended to a ZFC2 system containing an element corresponding to S. Formally, let $R, R', R'', ...$ etc range over ZFC2 systems, and let x^R abbreviate $R \upharpoonright x$. Then they should adopt the following principle.

(Capture) $$\Box\forall S\Box\forall R\Diamond\exists R' \sqsupseteq R\exists x \in R'(x^{R'} =^* S)$$

And it turns out that given Co-Realisability and Existence, Capture implies that Putnam-translation and ^{tr}-translation are equivalent.

Theorem 6. BT + Co-Realisability + Existence + Capture *proves* Stability*,* Extendibility*, and:*

$$\Box\forall R\forall\vec{x} \in R(\varphi^{pt}_R(\vec{x}) \leftrightarrow \varphi^{tr}(\vec{x^R}))$$

where $\varphi \in \mathcal{L}_\in$ *with free variables among* $\vec{x}$*.*

4.3 Relations

In this final section, I'll briefly discuss the question how exactly modal structuralists working in the framework of BT should make sense of relations.[63]

Let's start with Hellman [1996]'s proposal to understand relations mereologically. Roughly, the idea is that '$\langle x, y\rangle \prec xx$' should be taken to mean something like 'there is a fusion that behaves suitably like an ordered pair of x and y, and that fusion is among the xxs'. On this approach, Relation Stability would rely on something like the claim that the relevant fusions have the same parts in any possibility in which they exist. But is the modal structuralist entitled to this? The answer will clearly depend on their views on parthood. There are, roughly speaking, two camps. For some, parthood is of a piece with identity, and just as objects are usually thought to be identical or distinct as a matter of logical, and therefore metaphysical, necessity, objects have their parts as a matter of logical necessity, and therefore metaphysical, necessity.[64] For others, things can metaphysically, and therefore logically, change their parts. My hand, for example, could have failed to be a part of me. Many who take this view nevertheless think that in addition to objects for which parthood is metaphysically contingent, there are objects for which it's not. Indeed, some think that for any object, there's a co-incident object—an object with exactly the same parts—for which parthood is a metaphysically necessary matter.[65] Call such objects *metaphysically rigid*. By working with metaphysically rigid fusions, we could ensure Relation Stability for metaphysical possibility, but there seems to be no good reason to think that it will hold for logical possibility. And

[63]Some modal structuralists may prefer a more radical approach. Berry [2022], for example, prefers to avoid quantification in to modal contexts, and would thus be unhappy even working in the language of BT. Instead, she adds a new primitive modal operator meaning something like 'it is logically possible, keeping the structure of the extensions of predicates P, R, S,... etc fixed (up to isomorphism), that...', and thinks of a system as given by a 1-place predicate—whose extension at a given possibility is its domain—and a 2-place predicate—whose extension at a given possibility is its relation. With the right axioms governing the new modal operator, Berry [2022] is able to recover something like the Putnam-translations of the axioms of ZFC. It is a non-trivial task to transpose the sets as structures view articulated here into her framework.

[64]See, for example, [Armstrong, 1978, p.37-8]. Uzquiano [2014] is an in-depth discussion of the interaction between parthood and modality. In S5, the necessity of identity and distinctness are provable from the standard identity axioms. In weaker logics, the necessity of identity is typically provable, but the necessity of distinctness need not be. Some take this to be an important benefit of those logics (see, for example, Roberts [2021]).

[65]See Uzquiano [2014].

if the modal structuralist ignores the merely logically possible ZFC2 systems or set-systems, they could be charged with an arbitrary and unjustified curtailing of Cantor's paradise.

Regardless of the view we take on parthood, the mereological approach faces a version of the problem of control. In particular, the mereological assumptions required to obtain the relevant fusions are metaphysically non-trivial. If they don't hold as a matter of fact, then there's no guarantee that we will have, for example, a system of the actual electrons and their relation of entanglement.

Luckily, there are other approaches to relations. It's not hard to see that the results in this paper don't in any way depend on the assumption that relations are pluralities. Rather, they depend on the assumption that, whatever relations are, they satisfy versions of Relation Comp and Relation Stability. The results, in other words, would go through if we worked in the language of plural modal logic enhanced with a new sort for relations. Let me mention two approaches that exploit this fact. First, we could code ordered pairs of objects as higher-level pluralities.[66] In particular, given quantification over pluralities of pluralities—or *second-level* plural quantification—we could define ordered pairs of objects along the lines of the standard Kurasawa definition in set theory, but now using pluralities and second-level pluralities instead of sets. More precisely, if $xxx, yyy, zzz, ...$ etc range over second-level pluralities, then given the second-level analogue of Plural Comp and Plural Extensionality, we could prove that there is a second-level plurality corresponding to the Kurasawa ordered pair of any two objects:

(*) $$\forall x, y \exists! xxx \forall xx (xx \prec xxx \leftrightarrow \forall z(z \prec xx \leftrightarrow z = x) \vee \forall z(z \prec xx \leftrightarrow z = x \vee z = y))$$

A system would then comprise a plurality as the domain together with a plurality of pluralities of pluralities—or *third-level* plurality—as a relation. The analogue of Relation Stability would then follow from the higher-level versions of Plural Stability, and Relation Comp would follow from the higher-level versions of Plural Comp.

A second approach would be to understand relations in the framework of higher-order order logic as (second-order) relations. To ensure that Relation Stability comes out true on this account, we would need to work with *stable* relations: that is, relations R for which $\Box\forall x\Box\forall y\Box(R(x, y) \rightarrow \Box(ER, x, y \rightarrow R(x, y)))$. The analogue of Relation Comp for stable relations is not provable in standard versions of higher-order logic, but many higher-order theorists are happy to countenance it.

Of these different approaches to relations, my preference would be the higher-level pluralities approach. First, it relies on the same nothing over and above conception underlying Plural Comp, Plural Stability, and Plural Extensionality, but now extended to higher-level pluralities. This is unlike the mereological approach, where the genuinely new machinery of mereology is required. It is also unlike the higher-order logic approach, where we need genuinely new assumptions concerning the existence of stable relations. Although many higher-order theorists are indeed happy to endorse those assumptions, it's not obvious that they are entitled to them.[67] Second, because they following from the nothing over and above conception, the higher-level analogues of Plural Comp, Plural Stability, and Plural Extensionality plausibly hold for both metaphysical and logical possibility. If some things really are *nothing* over and above the individual things they comprise, then it's plausible to think that they couldn't even have logically existed without those individual things. Third, it's not subject to the problem of

[66]See Rayo [2006] and Linnebo [2017].
[67]See, for example, [Fritz, 2023, p.162].

control. The higher-level analogues of Plural Comp guarantee that there will always be a system whose domain comprises the electrons and whose relation coincides with entanglement on them, for example.

5 Technical appendix

In this section, I provide proofs for the main results of the paper. Much of the basic logical machinery is developed in Roberts [2019], which I will use here without comment. For example, in S5 one can move modalised claims freely in and out of modal contexts: e.g., $\Box(\varphi \wedge \Diamond\psi)$ is equivalent to $\Box\varphi \wedge \Diamond\psi$. Similarly, in positive free S5, we can reason with existential witnesses within the scope of modal operators: e.g., if φ (without x free) is provable from $\Diamond(\psi \wedge Ex)$, then it is provable from $\Diamond\exists x\psi$. I will use boldface variables $\mathbf{x}, \mathbf{y}, \mathbf{z}, ...$ etc as standing for either first-order or plural variables.

Definition 10. *Say that $\varphi \in \mathcal{L}^{\Box}$, with free variables among $\vec{\mathbf{x}}$, is (positively)* **stable** *if:*

$$\Box\forall\vec{\mathbf{x}}(\varphi \to \Box(E\vec{\mathbf{x}} \to \varphi))$$

and say that it is **strongly stable** *if:*

$$\Box\forall\vec{\mathbf{x}}(\Box\varphi \vee \Box\neg\varphi)$$

Definition 11. *Say that $\varphi \in \mathcal{L}^{\Box}$ is* **quasi-modalised** *if its quantifiers are either bounded—i.e. of the form $\exists x \prec xx$ or $\exists yy \subseteq xx$—or modalised—i.e. of the form $\Diamond\exists x$ or $\Diamond\exists xx$.*

Lemma 1 (BT)**.** *All quasi-modalised formulas in $\mathcal{L}$ are stable.*

Proof. See the proof of Lemma 5.20 in Roberts [2019]. $\Box$

It follows that the crucial notion of satisfaction is stable. This means the notion of being a set-system is also stable. Whenever a set-system exists, it is a set-system. It is also easy to see that $=^*$, $\in^*$, φ^{tr}, and co-realisability are all strongly stable.

We've been thinking of isomorphisms-as-pluralities. But, of course, it may be that in the course of an argument we are able to establish that a formula defines an isomorphism. For example, if S is a set-system with one element x in its domain and S' is a set-system with one element y in its domain, then "$x = x \wedge y = y$" defines an isomorphism between S and S'. Definable isomorphisms are valuable because they don't require that the isomorphic systems are compossible. The following lemmas tell us when we can move back and forth between isomorphisms-as-pluralities and such definable isomorphisms.

Definition 12. *Say that φ defines an isomorphism between S and S' when φ is strongly stable and:*

$$\Box\forall x \in S\Diamond(ES' \wedge \exists! y \in S'\varphi(x, y))$$

$$\Box\forall y \in S'\Diamond(ES \wedge \exists! x \in S\varphi(x, y))$$

$$\Box\forall x, y \in S\Box\forall x', y' \in S'(\varphi(x, x') \wedge \varphi(y, y') \to [\Diamond(ES \wedge S \vDash x \in y) \leftrightarrow \Diamond(ES' \wedge S' \vDash x' \in y')])$$

Lemma 2 (BT)**.** *Suppose φ defines an isomorphism between S and S'. Suppose, moreover, that $\Diamond(\vec{x} \in S), \Diamond(\vec{x'} \in S'), \Diamond(E\vec{xx}, S \wedge xx \vec{\subseteq} S), \Diamond(E\vec{xx}', S' \wedge xx' \vec{\subseteq} S'), \varphi(\vec{x, x'})$, and:*

$$\Box\forall y\Box(y \prec xx' \leftrightarrow \Diamond(y \in \vec{S'}) \wedge \Diamond\exists z \in S \cap xx(\varphi(z,y)))$$

Then:

$$\Diamond(ES \wedge S \vDash \psi(\vec{x}, \vec{xx})) \leftrightarrow \Diamond(ES' \wedge S' \vDash \psi(\vec{x'}, \vec{xx'}))$$

where $\psi \in \mathcal{L}^2_{\in}$ with free variables among $\vec{x}, \vec{X}$.

Proof. By a straightforward, but tedious, induction on the complexity of ψ. □

There is a simple connection between defined isomorphisms and isomorphisms as pluralities. Suppose that xx is a plurality coding an isomorphism between existing S and S'. Then the formula "$\Diamond(\langle x, y\rangle \prec xx)$" defines an isomorphism between them. Conversely, if φ defines an isomorphism between existing S and S', then $xx = \{\langle x, y\rangle : \varphi(x, y)\}$ will code an isomorphism between them. However, unlike possible pluralities coding isomorphisms, it turns out that any defined isomorphisms can be composed.

Definition 13. *If φ defines an isomorphism between S and S' and ψ defines an isomorphism between S' and S'', then we let $\psi \circ \varphi$ abbreviate the following formula:*

$$\Diamond\exists z \in S'(\varphi(x, z) \wedge \psi(z, y))$$

Lemma 3 (BT)**.** *If φ defines an isomorphism between S and S' and ψ defines an isomorphism between S' and S'', then $\psi \circ \varphi$ defines an isomorphism between S and S''.*

Proof. Straightforward. □

We are now in a position to prove one direction of the main theorem that ZF proves a sentence φ precisely when SAS proves φ^{tr}.

Theorem 7. *Let $\varphi \in \mathcal{L}_{\in}$ be a sentence. Then,* ZF *proves φ only if* SAS *proves φ^{tr}.*

Proof. Let E^*S abbreviate the claim that $\exists S'(S' = S)$ (that is, S exists and is a set-system).[68] We will show that when ZF proves φ, SAS proves:

$$\Diamond(\vec{E^*S}) \to \varphi^{tr}$$

where φ^{tr}'s free variables are among $\vec{S}$. The desired result will then follow, since $\Box\forall S\Diamond E^*S$ is trivially true in S5 and $\Box\forall S\varphi^{tr}$ entails φ^{tr} when S is not free in φ^{tr}, since $\Box\exists S(S = S)$ is trivially true—any plurality with one element and an empty relation will witness it.

There are two cases to consider. First, we need to show that the translations of logical axioms are provable in SAS and that the rules of inference are preserved under translation. Second, we need to show that the translation of each axiom of ZF is provable in SAS. Throughout I will assume that $\Diamond E^*S$, $\Diamond E^*S'$, and $\Diamond E^*S''$.

Case 1: Logic. Since the translation commutes with the connectives, the translations of propositional logical truths will be provable. For the classical quantifier axioms, we have:

[68] By Lemma 1, if $\Diamond E^*S$, then S will be a set-system whenever it exists.

Q1. ($\forall x\varphi \to \varphi[y/x]$, where y is free for x.) If $\Box\forall S\varphi^{tr}$ and $\Diamond E^*S'$, then $\Diamond\varphi^{tr}(S')$. It follows that $\varphi^{tr}(S')$ (since φ^{tr} is strongly stable).

Q2. ($\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$.) Trivial.

Q3. ($\forall x\varphi \leftrightarrow \varphi$, where x is not free in φ.) If $\Box\forall S\varphi^{tr}$, then φ^{tr} since φ^{tr} is strongly stable and $\Diamond\exists(S = S)$. Similarly, if φ^{tr}, then $\Box\varphi^{tr}$ and thus $\Box\forall S\varphi^{tr}$.

The classical identity axioms are a little trickier.

I1. ($x = x$.) Trivial.

We know that:

LL. $x = y \to (\varphi(x) \to \varphi(y))$

is provable from its atomic instances. So, we can focus on those. For atomic identity claims, symmetry is trivial. For transitivity: let $S =^* S'$ and $S' =^* S''$. We can take existential witnesses for various postulated systems and isomorphisms in these claims. We will then have a chain of systems $S, S_1, S_2, ..., S''$ and a sequence of formulas $\varphi_1, \varphi_2, ..., \varphi_n$ defining isomorphisms between adjacent pairs. By Lemma 3, that means there is a formula ψ defining an isomorphism between S and S''. Now, by Co-Realisability, there could be co-existing copies S^* and S^{**} of S and S'' respectively. So, we do the same move again and obtain a formula χ defining an isomorphism between S^* and S^{**}. But since S^* and S^{**} co-exist, Relation Comp will give us a plurality coding an isomorphism between them. Thus, $S^* \cong^\Diamond S^{**}$ and so $S =^* S''$, as required.

For the atomic membership claims, suppose that $S =^* S'$ and $S \in^* S''$. That means $S =^* S'' \restriction x$, for some $x < \mathsf{top}(S'')$. By the previous reasoning, it follows that $S' =^* S'' \restriction x$ too and so $S' \in^* S''$, as required. Now suppose that $S =^* S'$ and $S'' \in^* S$. That means $S'' =^* S \restriction x$, for some $x < \mathsf{top}(S)$. Now, by Lemma 3, there is a formula φ defining an isomorphism between S and S'. Let $xx \subseteq S'$ be the range of φ on the domain of $S \restriction x$. It is easy to check that within S, the domain of $S \restriction x$ is a transitive set-system. So, by Lemma 2, it follows that xx is a transitive set-system in S'. Now, it follows that xx together with S''s membership relation comprises a set-system S''', and it is easy to see that φ restricted to $S \restriction x$ defines an isomorphism between $S \restriction x$ and S'''. So, we can conclude from Lemma 3 and Co-Realisability that $S'' =^* S'''$ and so $S'' \in^* S'$ by the previous reasoning. Finally, suppose that $S \in^* S$. By Plural Comp, let $xx = \{x \in S : S \restriction x \cong S\}$. It is straightforward to check that xx is a non-well-founded subplurality of S, contradicting the fact that S satisfies $\mathsf{Foundation}_2$.

The classical rules of inference—modus ponens (MP) and universal generalisation (UG)—are easily seen to be preserved by the translation. For example, suppose we've proved $\Diamond\vec{E}^*S \wedge \Diamond E^*S' \to \varphi^{tr}$. Then we can infer $\Diamond\vec{E}^*S \wedge \Box\forall S'\Diamond E^*S' \to \Box\forall S'\varphi^{tr}$. Thus, since $\Box\forall S'\Diamond E^*S'$ is trivially true, we get our desired claim. It follows from all of this that we have:

(LL^{tr}) $$\Box\forall S\Box\forall S'(S =^* S' \wedge \varphi^{tr}(S) \to \varphi^{tr}(S'))$$

Case 1: Set Theory.

- EXTENSIONALITY. Suppose that S and S' are co-extensive^{tr}. By Co-Realisability, let S'' and S''' be compossible copies of S and S'. LL^{tr} implies that S'' and S''' are co-extensive^{tr}. We'll show that whenever ES'', S''', they're isomorphic and thus that $S'' =^* S'''$. It will then follow from LL^{tr} that $S =^* S'$ as required. So, suppose ES'', S'''. Let $x < \mathsf{top}(S'')$. Then, since S'' and S''' are co-extensive^{tr}, it follows that $S'' \restriction x =^* S''' \restriction y$ for some $y < \mathsf{top}(S''')$. Thus, $S'' \restriction x \cong S''' \restriction y$. A simple induction on S''' shows that for any $z, z' \in S'''$, if $S''' \restriction z \cong S''' \restriction z'$, then the witnessing isomorphism must be the identity. It follows that both y and the isomorphism witnessing $S'' \restriction x \cong S''' \restriction y$ are unique (up to extension). By Relation Comp , let xx be the union of all those isomorphisms. A simple induction on S'' shows that xx is the required isomorphism between S'' and S'''.

- SEPARATION. Suppose ES. By Plural Comp, let $xx = \{x < \mathsf{top}(S) : \varphi^{tr}(S \restriction x)\}$. Let S' be S modified so that its top element is above all and only the xxs. Then we show that S' contains* all and only the possible φ^{tr}s in* S. Suppose $\varphi^{tr}(S'')$ and $S'' \in^* S$. Then, $S'' =^* S \restriction x$ for some $x < \mathsf{top}(S)$. It follows from LL^{tr} that $\varphi^{tr}(S \restriction x)$ and so $x \prec xx$ and $S \restriction x \in^* S'$. Thus, $S'' \in^* S'$ by LL^{tr}. Conversely, suppose $S'' \in^* S'$. Then $S'' =^* S' \restriction x$ for some $x < \mathsf{top}(S')$. So, $x \prec xx$ and $S' \restriction x = S \restriction x$ where $\varphi^{tr}(S \restriction x)$. So, $\varphi^{tr}(S'')$ and $S'' \in^* S$ by LL^*.

- UNION. Suppose ES. By Plural Comp, let $xx = \{x \in S : \exists y < \mathsf{top}(S)(S \vDash x \in y)\}$. Let S' be S modified so that its top element is above all and only the xxs. Then we show that S' contains* all and only the possible elements* of elements* of S. Suppose $S'' \in^* S''' \in^* S$. Then, $S''' =^* S \restriction x$ for some $x < \mathsf{top}(S)$. It follows from LL^{tr} that $S'' \in S \restriction x$. Thus, $S'' =^* S \restriction x \restriction y$ for some $y < \mathsf{top}(S \restriction x)$. Since $y \prec xx$ and $S \restriction x \restriction y = S \restriction y = S' \restriction y$, $S'' \in^* S'$ as required. Conversely, suppose $S'' \in^* S'$. Then $S'' =^* S' \restriction x$ for some $x < \mathsf{top}(S')$. Thus, $S' \restriction x \in^* S \restriction y$ for some $y < \mathsf{top}(S)$ and so $S'' \in^* S \restriction y \in^* S$ by LL^*.

- FOUNDATION. Suppose ES, that S is non-empty, and that every element* of S has non-empty* intersection* with S. By Plural Comp, let $xx = \{x \in S : S \restriction x \in^* S\}$. It is straightforward to check that xx is a non-well-founded subplurality of S, contradicting the fact that S satisfies Foundation$_2$.

- INFINITY. Infinity^{tr} is an immediate consequence of Infinity.

- PAIRING, POWERSET, AND COLLECTION. Powerset^{tr} and Collection^{tr} are immediate consequences of Collection, given Separation^{tr}. Pairing^{tr} follows from Co-Realisability given Infinity^{tr} and Separation^{tr}.

□

Theorem 8. *Let $\varphi \in \mathcal{L}_\in$ be a sentence. If φ^{tr} is provable in* SAS*, then φ is provable in* ZF.

Proof. Let $\mathcal{L}_\in^{\langle x,y\rangle}$ be the definitional expansion of $\mathcal{L}_\in$ with terms $\langle x, y\rangle$ for ordered pairs. We then define the following translation from $\mathcal{L}$ to $\mathcal{L}_\in^{\langle x,y\rangle}$. Let $w, w', w'', ...$ etc range over transitive sets closed under ordered pairs.

- $(x = y)^\dagger_w = (x = y)$, $(xx = yy)^\dagger_w = (z = v)$

- $(x \prec xx)^\dagger_w = (x \in z)$, $(\langle x, y\rangle \prec xx)^\dagger_w = (\langle x, y\rangle \in z)$
- ${}^\dagger_w$ commutes with the connectives.
- $(\exists x\varphi)^\dagger_w = \exists x \in w\varphi^\dagger_w$, $(\exists xx\varphi)^\dagger_w = \exists z \subseteq w\varphi^\dagger_w$
- $(\Diamond\varphi)^\dagger_w = \exists w'\varphi^\dagger_{w'}$

It is easy to see that ZF proves the †-translations of all the theorems of SAS. Moreover, a simple induction on the complexity of $\varphi \in \mathcal{L}_\in$ shows that when $\vec{x} \in w$:

$$(\varphi^{tr})^\dagger \leftrightarrow \varphi^*$$

Thus, by the folklore result I mention earlier:

$$(\varphi^{tr})^\dagger \leftrightarrow \varphi$$

It follows that when SAS proves φ^{tr}, ZF proves φ, as required. □

Theorem 9. BT + Existence + Co-Realisability + Capture *proves* Extendibility, Stability, *and:*

$$\Box\forall R\forall\vec{x} \in R(\varphi^{pt}_R(\vec{x}) \leftrightarrow \varphi^{tr}(\vec{x^R}))$$

where $\varphi \in \mathcal{L}_\in$ with free variables among $\vec{x}$.

Proof. Working in BT + Existence + Co-Realisability + Capture, let $R, R', R'', ...$ etc range over ZFC2 systems, and let x^R abbreviate $R \upharpoonright x$. By induction on the complexity of φ, we first prove that:

$$\Box\forall R\forall\vec{x} \in R(\varphi^{pt}_R(\vec{x}) \leftrightarrow \varphi^{tr}(\vec{x^R}))$$

Let ER be a ZFC2 system and let $\vec{x} \in R$. For the atomic membership case: assume $R \vDash x \in y$. Then $x^R \in^* y^R$. Conversely, suppose $x^R \in^* y^R$. Then $x^R =^* y^R \upharpoonright z$ for some $z < \mathsf{top}(y^R)$. It follows from Lemma 3 that $x^R \cong z^R$ for $R \vDash z \in y$. Since x^R and z^R are transitive in R, their isomorphism will be the identity. Thus, $R \vDash x \in y$. For the atomic identity case: if $x = y$, then trivially $x^R =^* y^R$. Conversely, if $x^R =^* y^R$, then $x^R \cong y^R$ by Lemma 3, and since x^R and y^R are transitive in R, their isomorphism will be the identity.

Since both translations commute with the connectives, the conjunction and negation cases are trivial. For the existential case: assume $\Diamond\exists R' \sqsupseteq R\exists x \in R'\varphi^{pt}_{R'}$. By Plural Stability, $\vec{x}$ will exist in this possibility and will be in R'. So, we can apply the induction hypothesis there to conclude that $\Diamond\exists S\varphi^{tr}$, as required. Now suppose that $\Diamond\exists S\varphi^{tr}$. So, suppose $\Diamond(E^*S \wedge \varphi^{tr})$ is a witness. By Capture, $\Diamond\exists R' \sqsupseteq R\exists x \in R'(x^{R'} =^* S)$. By Plural Stability, $\vec{x}$ will exist in this possibility and will be in R'. By Lemma 1, φ^{tr} will be true there too. So, it follows from LL^{tr}—which was provable in BT + Co-Realisability alone—that $\varphi^{tr}(x^{R'})$ there. Thus, by the induction hypothesis there, it follows that $\varphi^{pt}_{R'}$ there, and thus that $\Diamond\exists R' \sqsupseteq R\exists x \in R'\varphi^{pt}_{R'}$.

Next, we use Existence to prove $\varphi^{pt} \leftrightarrow \varphi^{tr}$ when φ is a sentence. Without loss of generality, we can assume φ is of the form $\exists x\psi$. So, suppose $\Diamond\exists R\exists x \in R\psi^{pt}_R$. Then, by the previous result, we have $\Diamond\exists R\exists x \in R\psi^{tr}(x^R)$ and thus φ^{tr} by Lemma 1, as required. Now suppose that $\Diamond(ES \wedge \psi^{tr}(S))$. By Capture and Existence, $\Diamond\exists R\exists x \in R(x^R =^* S)$ and thus by the previous result, Lemma 1, and LL^{tr}, $\Diamond\exists R\exists x \in R\psi^{pt}_R$, which is to say φ^{pt}. Since Stability comprises the Putnam-translations of instances of the vacuous quantifier axiom, it follows immediately.

Finally, to prove Extendibility, first take any possible R and modify it by removing one element x to obtain an isomorphic R'. By adding x as a top element to R' we get a set-system S. By Capture, $\Diamond\exists R'' \sqsupseteq R \exists y \in R''(y^{R''} =^* S)$. Since S is equinumerous with R, it follows that R'' must be a proper end-extension. □

It footnote 32, I claimed that given some natural background assumptions, theorem 3 can be proved in weaker modal logics than S5 given NNE—and its extensions to pluralities—and the G principle of modal logic. Let me now make that claim more precise.

The new background assumptions are as follows. First, we have the necessity of distinctness (where **x** is either a first-order or plural variable).

(Stab$_{\neq}$) $$\forall \mathbf{x}, \mathbf{y}(\mathbf{x} \neq \mathbf{y} \to \Box(\mathbf{x} \to \mathbf{y}))$$

Next, we have negative stability for the relations '$x \prec xx$' and '$\langle x, y\rangle \prec xx$'.

(Stab$_{\not\prec}$) $$\forall x, xx(x \not\prec xx \to \Box(x \not\prec xx))$$

(Stab$_{\langle\rangle\not\prec}$) $$\forall x, y, xx(\langle x, y\rangle \not\prec xx \to \Box(\langle x, y\rangle \not\prec xx))$$

Finally, we have inextensibility principles for pluralities, which say that we can add neither elements nor subpluralities to given pluralities.

(Inext$_{\prec}$) $$\forall xx(\Diamond\exists x \prec xx\varphi \to \exists x \prec xx\Diamond\varphi)$$

(Inext$_{\subseteq}$) $$\forall xx(\Diamond\exists yy \subseteq xx\varphi \to \exists yy \subseteq xx\Diamond\varphi)$$

Both principles can be seen to follow from the nothing over and above conception of pluralities (see Roberts [2022], especially footnote 26).

Now, let NNE* be the conjunction of NNE with its extension to pluralities, let S4.2 denote, as usual, S4 + G, and let $\mathsf{MSST}^-_{\mathsf{S4.2+NNE^*}}$ be the theory, over a positive free S4.2 + NNE* modal logic, comprising the axioms Stab$_{\neq}$, Stab$_{\not\prec}$, Stab$_{\langle\rangle\not\prec}$, Inext$_{\prec}$, Inext$_{\subseteq}$; the axioms Plural Comp, Plural Stability, Plural Extensionality, Relation Comp, and Relation Stability; and Existence and Extendibility. Then I claim that we have the following result.

Claim 2. *$\mathsf{MSST}^-_{\mathsf{S4.2+NNE^*}}$ proves φ^{pt} if and only if Z^* + In proves φ.*

It follows immediately from claim 2 that $\mathsf{MSST}^-_{\mathsf{S4.2+NNE^*}}$ proves Stability. I omit the proof of claim 2 here, but it's not hard to reconstruct it on the basis of the proof of theorem 3 in Roberts [2019], using obvious modifications for the new modal logic and the new background principles.

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